UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 1.30 to 4.30

PAPER 63

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Information

The signature is $(+ - - -)$, *and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

You may use freely any information on the lecture handout on linearized perturbations included with this examination paper.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Lecture Handout

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A *perfect fluid* with energy density ρ, pressure p (both measured in the rest frame) and 4-velocity U^a (normalized so that $U_a U^a = 1$) has energy momentum tensor

$$T^{ab} = (\rho + p)U^a U^b - p g^{ab}.$$

Appealing to a conservation law derive the relativistic Euler equation

$$(\rho + p)U_{a;b}U^b = p_{,a} - U_a p_{,b} U^b.$$

Now assume hydrostatic equilibrium obtains i.e.,

(a) There exists a timelike Killing vector K^a such that $\mathcal{L}_K g_{ab} = \mathcal{L}_K \rho = \mathcal{L}_K p = 0$.

(b) The velocity U^a is proportional to K^a.

Express U^a in terms of K^a and hence show that

$$p_{,a} = -(\rho + p)(\log(K_c K^c)^{1/2})_{,a}.$$

Next consider a perfect fluid made up of black body radiation, i.e., $p = \frac{1}{3}\rho$. Show that in hydrostatic equilibrium

$$\rho \propto (K_c K^c)^{-2}.$$

Deduce that in a regular spacetime a compact body in hydrostatic equilibrium made from such fluid cannot have a free surface, i.e., one where $\rho \to 0$.

2 Consider the following Newtonian experiment in a terrestrial laboratory of dimension L. Two particles of equal mass m are placed one above the other with vertical separation ζ_0 at $t = 0$ and are then allowed to fall freely. Given that the Newtonian gravitational potential is $\phi = -GM/z$, where G is Newton's constant, M is the mass of the Earth and z is the distance to the centre of the Earth, justify the following equation for the vertical separation $\zeta(t)$,

$$\ddot{\zeta} = \frac{2GM}{z^3}\zeta.$$

Now suppose that one particle carries charge q, the other is uncharged and there is a uniform vertical electric field of magnitude E. What is the additional contribution to $\ddot{\zeta}$? Assuming that L is sufficiently small for variations in GM/z^3 and E to be neglected, estimate, roughly, the changes in $\zeta(t)$ due to gravitational and electric effects respectively in a laboratory experiment. Demonstrate that in the limit $L \to 0$ gravitational effects can be neglected.

Use this result to write an essay on the *strong equivalence principle* and its implications for a relativistic theory of gravity.

3 Suppose that in Minkowski spacetime with cartesian coordinates the symmetric tensor $T^{ab}(t, \mathbf{x})$ has compact support and $T^{ab}{}_{,b} = 0$. Show that

$$\int_\Omega d^3x\, T^{\alpha\beta}(t, \mathbf{x}) = \frac{1}{2}\left(\frac{d}{dt}\right)^2 \int_\Omega d^3x\, T^{00}(t, \mathbf{x}) x^\alpha x^\beta,$$

where Ω denotes all 3-space.

Now consider tensor perturbations away from Minkowski spacetime of Einstein's field equations $G^{ab} = -8\pi G T^{ab}$ for a source with compact support. Show, using the notation of the lecture handout, that far from the source and to leading order

$$E^{\alpha\beta}(t, \mathbf{x}) = \frac{G}{r} \ddot{Q}^{\alpha\beta}(t - r),$$

where r and $Q^{\alpha\beta}$ should be identified.

[*Hint: You may assume that the retarded potential solution of* $\Box\, u(t, \mathbf{x}) = f(t, \mathbf{x})$ *is*

$$u(t, \mathbf{x}) = \frac{1}{4\pi} \int_\Omega d^3x' \, \frac{f(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$

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Two stars each of mass m move in a circle of radius R in the xy-plane centred on the origin under their mutual, Newtonian, attraction. Show that their positions may be taken to be

$$\mathbf{x} = \pm(R\cos\theta(t), R\sin\theta(t), 0),$$

where $\dot{\theta}^2 = Gm/(4R^3)$. Show that to leading order

$$Q^{\alpha\beta}(t) = mR^2 \begin{pmatrix} \cos 2\theta + \frac{1}{3} & \sin 2\theta & 0 \\ \sin 2\theta & \frac{1}{3} - \cos 2\theta & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}.$$

Compute and interpret $E^{\alpha\beta}(t, \mathbf{x})$.

4 Consider the following line element for 2-dimensional de Sitter spacetime

$$ds^2 = dt^2 - \cosh^2 t \, d\chi^2 \,.$$

What are the ranges of t and χ? Obtain the equations governing geodesics in this spacetime and sketch all of the geodesics through a fixed point.

Obtain a conformal compactification of this spacetime and use it to discuss the horizon structure.

[*The transformation*

$$T = 2\tan^{-1}(e^t) - \tfrac{1}{2}\pi,$$

may prove useful.]

END OF PAPER