MATHEMATICAL TRIPOS Part III

Monday 2nd June 2008 9am to 11 am

PAPER 61

CONTROL OF QUANTUM SYSTEMS

Attempt 3 of the following four questions. All questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Controllability and Interaction Picture.

(a) Briefly explain in about one sentence each the general meaning of the notions of reachable sets and controllability in control theory, and explain how these notions can be applied to (Hamiltonian) quantum control systems, e.g., in the context of quantum state or process engineering.

(b) Consider the bilinear Hamiltonian control system given by $\hat{H}[f(t)] = \hat{H}_0 + f(t)\hat{H}_1$, where

$$\hat{H}_0 = \begin{pmatrix} -\omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\omega_2 \end{pmatrix}, \quad \hat{H}_1 = \begin{pmatrix} 0 & d_1 & 0 \\ d_1 & 0 & d_2 \\ 0 & d_2 & 0 \end{pmatrix} \tag{1}$$

and $\omega_1, \omega_2, d_1, d_2 \in \mathbb{R}$. Explain briefly whether this system controllable if $\omega_k > 0$, $d_k > 0$, for $k = 1, 2$ and $\omega_1 \neq \omega_2$. You may use theorems from the course without proof but be explicit about the notion of controllability you are using.

(c) Show that the system considered in part (b) is *not* controllable (in any sense) if $\omega_1 = \omega_2 = \omega$ and $d_1 = d_2 = d$. **Hint:** Show that $|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$ is a dark state, i.e., that $|-\rangle$ is an eigenstate of $\hat{H}[f(t)]$ for any $f(t)$ and consider what this means for the set of states reachable from $|-\rangle$.

(d) Let $\hat{U}(t)$ be the solution of the Schrodinger equation $i\hbar\frac{d}{dt}\hat{U}(t) = \hat{H}[f(t)]\hat{U}(t)$ with $\hat{U}(0) = \hat{I}$ and $H[f(t)] = \hat{H}_0 + f(t)\hat{H}_1$ as above. Show that the interaction picture evolution operator $\hat{U}_I(t) = \hat{U}_0(t)^\dagger\hat{U}(t)$ with $\hat{U}_0(t) = \exp(-it\hat{H}_0/\hbar)$ satisfies

$$i\hbar\frac{d}{dt}\hat{U}_I(t) = f(t)\hat{H}_I(t)\hat{U}_I \qquad \text{with} \qquad \hat{H}_I(t) = \hat{U}_0(t)^\dagger\hat{H}_1\hat{U}_0(t). \tag{2}$$

(e) Assuming $\hat{H}_0$ and $\hat{H}_1$ as in Eq. (1) and choosing units such that $\hbar = 1$ for convenience, show that the interaction picture Hamiltonian is

$$\hat{H}_I(t) = \begin{pmatrix} 0 & d_1 e^{-i\omega_1 t} & 0 \\ d_1 e^{i\omega_1 t} & 0 & d_2 e^{i\omega_2 t} \\ 0 & d_2 e^{-i\omega_2 t} & 0 \end{pmatrix}. \tag{3}$$

(f) Assume $f(t) = A_1(t)\cos(\omega_1 t)$ and $\Delta\omega = \omega_2 - \omega_1$. Using the interaction picture Hamiltonian (3), show that

$$f(t)\hat{H}_I = \frac{A_1(t)}{2}\left[\begin{pmatrix} 0 & d_1 & 0 \\ d_1 e^{i2\omega_1 t} & 0 & d_2 e^{i(2\omega_1+\Delta\omega)t} \\ 0 & d_2 e^{-i\Delta\omega t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & d_1 e^{-i2\omega_1 t} & 0 \\ d_1 & 0 & d_2 e^{i\Delta\omega t} \\ 0 & d_2 e^{-i(2\omega_1+\Delta\omega)t} & 0 \end{pmatrix}\right]. \tag{4}$$

Explain under what assumptions can we simplify

$$f(t)\hat{H}_I \approx \frac{A_1(t)d_1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

2 Geometric and Adiabatic Control.

(a) Let A be an operator in a finite dimensional Hilbert space $\mathcal{H}$ satisfying $A^2 = I$, where I is the identity operator. Show that $\exp(-i\theta A) = \cos(\theta)I - i\sin(\theta)A$ for $\theta \in \mathbb{R}$.

(b) Use the result from part (a) to show that

$$\exp(-i\theta A) = \begin{pmatrix} \cos\theta & -ie^{-i\phi}\sin\theta \\ -ie^{i\phi}\sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

for $A = \cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y$, where $\phi \in \mathbb{R}$ and the Pauli matrices are as usual

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

(c) Let $\hat{H}_k(\phi) = \cos(\phi)\hat{x}_k + \sin(\phi)\hat{y}_k$ for $k = 1, 2$, where

$$\hat{x}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{y}_1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{x}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{y}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Using the results from (b) explain why

$$\hat{U}_1(t; \Omega_1, \phi_1) = \exp_+\left[-\frac{i}{\hbar}\int_0^t \Omega_1(\tau)\,\hat{H}_1(\phi_1)\,d\tau\right] = \begin{pmatrix} \cos\theta_1(t) & -ie^{-i\phi_1}\sin\theta_1(t) & 0 \\ -ie^{i\phi_1}\sin\theta_1(t) & \cos\theta_1(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8a}$$

$$\hat{U}_2(t; \Omega_2, \phi_2) = \exp_+\left[-\frac{i}{\hbar}\int_0^t \Omega_2(\tau)\,\hat{H}_2(\phi_2)\,d\tau\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2(t) & -ie^{-i\phi_2}\sin\theta_2(t) \\ 0 & -ie^{i\phi_2}\sin\theta_2(t) & \cos\theta_2(t) \end{pmatrix} \tag{8b}$$

for $\theta_k(t) = \frac{1}{\hbar}\int_0^t \Omega_k(\tau)\,d\tau$, where $\exp_+$ indicates positive time ordering. Explain whether

$$\exp_+\left[-\frac{i}{\hbar}\int_0^t \Omega_1(\tau)\,\hat{H}_1(\phi_1) + \Omega_2(\tau)\hat{H}_2(\phi_2)\,d\tau\right] = \exp\left[-i\theta_1(t)\hat{H}_1(\phi_1) + \theta_2(t)\hat{H}_2(\phi_2)\right] \tag{9}$$

is true or not.

(d) Consider the drift-free bilinear Hamiltonian control system

$$\hat{H}(\Omega_k, \phi_k) = \Omega_1(t)\hat{H}_1(\phi_1) + \Omega_2(t)\hat{H}_2(\phi_2) \tag{10}$$

with $\hat{H}_k(\phi_k)$ as defined in (c), where the control variables are the pulse envelopes $\Omega_k(t)$ and phases ϕ_k for $k = 1, 2$. Using the expressions (8a) and (8b), explain how we can implement the following gates

$$\hat{W}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{W}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \hat{W}_4 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{11}$$

using simple geometric pulses or pulse sequences. If the system is initially in the state the eigenstate $|1\rangle \equiv (1, 0, 0)^T$, how can we transfer the population to the state $|3\rangle \equiv (0, 0, 1)^T$ using two simple pulses?

(e) If the pulse phases $\phi_1 = \phi_2 = 0$ then the Hamiltonian in part (d) simplifies

$$\hat{H}(\Omega_k) = \hat{H}(\Omega_k, \phi_k = 0) = \begin{pmatrix} 0 & \Omega_1(t) & 0 \\ \Omega_1(t) & 0 & \Omega_2(t) \\ 0 & \Omega_2(t) & 0 \end{pmatrix} \tag{12}$$

and we can show that $|\Psi(\theta_t)\rangle = \cos\theta_t|1\rangle - \sin\theta_t|3\rangle$ with $\theta_t = \arctan\left(\frac{\Omega_1(t)}{\Omega_2(t)}\right)$ is an eigenstate of $\hat{H}(\Omega_k)$ with eigenvalue $\lambda_0 = 0$.

(i) Explain briefly how we can exploit the fact that $|\Psi(\theta_t)\rangle$ is an eigenstate of $\hat{H}(\Omega_k)$ to adiabatically transfer population from the initial state $|1\rangle$ to the target state $|3\rangle$ (modulo global phases) by varying $\Omega_1(t)$ and $\Omega_2(t)$.

(ii) Assuming the intermediate level $|2\rangle$ is an excited state prone to decay by spontaneous emission, what is the main advantage of the adiabatic transfer scheme versus the geometric pulse sequence derived in part (d)?

3 Question 3. Optimal and Adaptive Control

(a) One approach in quantum control is to formulate the control problem as an optimization problem by choosing an objective functional to be maximized or minimized, and trying to find a control from a set of admissible controls for which the target functional assumes its minimum or maximum. Give possible objective functions for the following control problems:

(i) Maximizing expectation value of an observable $\hat{A}$ at a target time t_F for a pure-state Hamiltonian system.

(ii) Minimizing the gate error of a (unitary) process $\hat{U}_T$ with gate operation time T.

(iii) Steering the system to a target state $\hat{\rho}_d$ in time t_F.

Briefly justify your choice in each case, and explain the meaning of the symbols you use.

(b) Explain how we can find controls that maximize the objective function experimentally using adapative open-loop control, assuming we can repeat individual experiments with different controls and experimentally evaluate the objective function. You may wish to mention several algorithmic approaches but it suffices to describe one.

(c) If we are dealing with a complex system, for which we do not have an accurate mathematical model, then direct laboratory optimization using the adaptive/learning control approach discussed in part (b) is sometimes the only feasible way to find a control. However, if we have a model for the dynamics of the system, including its free evolution and the effect of external control fields, then there are more efficient optimization techniques. Describe one approach based on a variational formulation and numerical solution of the Euler-Lagrange equations.

(d) Suppose, using the technique in part (c), that we have found a complicated temporal pulse shape $f(t)$ that optimizes our target functional subject to constraints, etc. Sketch one way such a complex pulse shape could be realized experimentally using spectral pulse shaping. A sketch of a possible experimental setup and brief explanation will suffice.

4 Question 4. Feedback control

(a) Let $\hat{\rho}(t)$ and $\hat{\rho}_d(t)$ be density operators acting on a finite-dimensional Hilbert space $\mathcal{H}$ satisfying

$$\dot{\hat{\rho}}(t) = [-i(\hat{H}_0 + f(t)\hat{H}_1), \hat{\rho}(t)], \quad \dot{\hat{\rho}}_d(t) = [-i\hat{H}_0, \hat{\rho}_d(t)], \tag{13}$$

respectively, where $\hat{H}_0$ and $\hat{H}_1$ are Hermitian operators on $\mathcal{H}$ and $f(t)$ is a real-valued function. Show that the distance of the system state $\hat{\rho}(t)$ from the target state $\hat{\rho}_d(t)$ is monotonically decreasing if we choose $f(t) = \mathrm{Tr}(\hat{\rho}_d(t)\hat{\rho}(t))$.

Hint: Show $\dot{V}(t) \leq 0$ for $V(\hat{\rho}(t), \hat{\rho}_d(t)) = ||\hat{\rho}(t) - \hat{\rho}_d(t)||^2$ where $||x|| = \sqrt{\mathrm{Tr}(x^\dagger x)}$ is the Hilbert-Schmidt norm, observing that $\mathrm{Tr}([-i\hat{H}_0, \hat{\rho}_d(t)]\hat{\rho}(t)) = -\mathrm{Tr}(\hat{\rho}_d(t)[-i\hat{H}_0, \hat{\rho}(t)])$.

(b) The scheme in part (a) essentially provides a feedback control law that steers the system from some initial state $\hat{\rho}(0)$ to a desired target state $\hat{\rho}_d(t)$. Could this feedback law be used for measurement-based feedback control for quantum systems? If not, why not?

(c) In the standard semi-classical model of quantum control the goal is to control a quantum system using external fields produced by essentially classical actuators and measurements. An alternative approach is to replace the classical controller by another quantum system that acts as a quantum controller. A very simple example of such a system is a cavity that interacts with a quantized external field. Let $\hat{b}_0$, $\hat{b}_1$ and $\hat{a}$ be stochastic operators representing the input, output and cavity mode, respectively. It can be shown that for a simple cavity with cavity decay rate γ we obtain the following linear control system

$$\frac{d}{dt}\hat{a}(t) = -\frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{b}_0(t) \tag{14a}$$

$$\hat{b}_1(t) = \sqrt{\gamma}\hat{a}(t) + \hat{b}_0(t). \tag{14b}$$

By taking the Laplace transform of the equations, show that $\tilde{b}_1(s) = G(s)\tilde{b}_0(s)$ with gain function $G(s) = \frac{s-\gamma/2}{s+\gamma/2}$, where $\tilde{b}_0(s) = L[\hat{b}_0(t)](s)$ and $\tilde{b}_1(s) = L[\hat{b}_1(t)](s)$ are the Laplace transforms of $\hat{b}_0(t)$ and $\hat{b}_1(t)$, respectively, and apply Nyquist's stability criterion to decide if the cavity-field system is stable.

Hint: The Laplace transform L is linear and satisfies $L[\frac{d}{dt}a(t)](s) = sL[a(t)](s)$ assuming $a(0) = 0$.

(d) Consider the closed-loop system consisting of a cavity and a beamsplitter as pictured below. Using the state equations for the cavity (14) from part (c) and the input-output relation for the beamsplitter

$$\hat{b}_0 = \beta\hat{b}_{in} - \alpha\hat{b}_1 \tag{15a}$$

$$\hat{b}_2 = \alpha\hat{b}_{in} + \beta\hat{b}_1 \tag{15b}$$

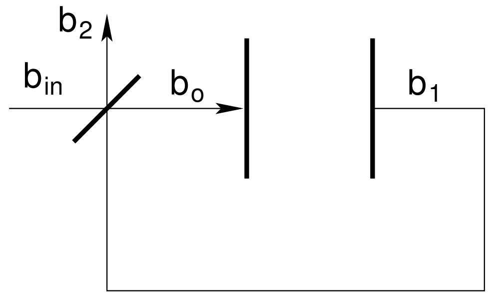
where α, β real with $\alpha^2 + \beta^2 = 1$, show that

$$\frac{d}{dt}\hat{a} = \frac{\gamma(\alpha - 1)}{2(1+\alpha)}\hat{a} - \frac{\beta\sqrt{\gamma}}{1+\alpha}\hat{b}_{in} \tag{16a}$$

$$\hat{b}_0 = \frac{\beta}{1+\alpha}\hat{b}_{in} - \frac{\alpha\sqrt{\gamma}}{1+\alpha}\hat{a} \tag{16b}$$

$$\hat{b}_1 = \frac{\beta}{1+\alpha}\hat{b}_{in} + \frac{\sqrt{\gamma}}{1+\alpha}\hat{a} \tag{16c}$$

$$\hat{b}_2 = \hat{b}_{in} + \frac{\beta\sqrt{\gamma}}{1+\alpha}\hat{a}. \tag{16d}$$



Cavity with feedback via beamsplitter.

(e) Noting that the transfer function for the closed-loop system in part (d) is $M(s) = \frac{\beta G(s)}{1+\alpha G(s)}$, where $G(s)$ is the cavity gain function from part (c), show that the closed-loop system is stable and calculate its steady state as a function of the input $\hat{b}_{in}$.

END OF PAPER