MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 1.30 to 4.30

PAPER 60

MATHEMATICAL ASPECTS OF QUANTUM INFORMATION THEORY

Attempt no more than **FOUR** *questions.*

There are **FIVE** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe a memoryless classical information source. What is meant by the ϵ-typical set of sequences for such a source?

State the Typical Sequence Theorem and use it to prove that for a memoryless classical information source with Shannon entropy $H(U)$, there exists a reliable compression-decompression scheme of rate R, for any $R > H(U)$. Describe such a compression-decompression scheme.

Consider a memoryless quantum information source which emits the states

$$|0\rangle = (1\ 0)^T \quad \text{or} \quad |+\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T,$$

with equal probability. Here T denotes transposition. What is the optimal rate of data compression for such a source?

2 How would you characterize an error operator acting on n qubits? What are the assumptions underlying this characterization?

Let $|\underline{x}\rangle$ denote a basis vector in the n-qubit Hilbert space, $\mathcal{H}^{\otimes n}$. How would you describe the action of an error operator on $|\underline{x}\rangle$?

Define the weight of a Pauli operator and the distance of a quantum code. Prove that an $[[n, k, d]]$ quantum code, $\mathcal{X}$, detects $d-1$ errors, and corrects $\lfloor (d-1)/2 \rfloor$ errors.

What are the basis codewords of the $[[9, 1]]$ Shor Code? How would you use this code to diagnose and correct a single phase flip error on the fourth qubit ?

3 Define the relative entropy, $D(p||q)$, of two probability distributions $p = \{p(x)\}_{x\in J}$ and $q = \{q(x)\}_{x\in J}$, where J is a finite alphabet, specifying any conventions that you use. Prove that $D(p||q) \geq 0$, carefully justifying your steps.

Let ρ and σ be two density matrices and let Φ be a completely positive trace-preserving (CPT) map. If

$$\rho' = \Phi(\rho) \text{ and } \sigma' = \Phi(\sigma),$$

state how the quantum relative entropies $S(\rho'||\sigma')$ and $S(\rho||\sigma)$ are related to each other.

Consider the following density matrices in the Hilbert space $\mathbf{C}^n \otimes \mathcal{H}$:

$$\rho = \sum_{i=1}^{n} p_i |i\rangle\langle i| \otimes \rho_i \text{ and } \sigma = \sum_{i=1}^{n} p_i |i\rangle\langle i| \otimes \sigma_i.$$

Here ρ_i, σ_i are density matrices in $\mathcal{H}$, and $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$. Evaluate the quantum relative entropy $S(\rho||\sigma)$ for these density matrices. Use the result to prove the joint convexity:

$$S(\sum_{i=1}^{n} p_i\rho_i || \sum_{i=1}^{n} p_i\sigma_i) \leq \sum_{i=1}^{n} p_i S(\rho_i||\sigma_i).$$

[*Hint:* Note that taking a partial trace is a CPT map].

Consider a bipartite quantum system, AB, which is in a state ρ_{AB}. Define its mutual information $S(A:B)$ and prove that it can be expressed as a relative entropy. Justify why $S(A:B)$ can never be negative.

4 Let M be an element of the n-qubit Pauli group $\mathbb{G}_n$. Let $\mathbb{S}$ denote the stabilizer of a code $\mathcal{X}_{\mathbb{S}}$. How is it related to $\mathbb{G}_n$? Define the code $\mathcal{X}_{\mathbb{S}}$ associated with $\mathbb{S}$.

Prove that each non-trivial element of $\mathbb{S}$ has two eigenvalues of equal multiplicities, carefully justifying all your steps.

What are the generators of $\mathbb{S}$? How many generators does a code $\mathcal{X}_{\mathbb{S}}$ of dimension 2^k have ?

Explain how a non-degenerate $\mathcal{E}$-correcting stabilizer code, $\mathcal{X}_{\mathbb{S}}$, can detect and correct an error characterized by an operator $E_{\underline{\alpha}}$. When would a stabilizer code fail to detect an error characterized by an operator $E_{\underline{\alpha}}$?

The $[[3, 1]]$ quantum repetition code is a stabilizer code. What are its generators? Why can it not correct a phase flip error?

5 Prove that a map Φ is completely positive and trace preserving (CPT) if and only if

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger,$$

for any density matrix ρ, where $\{A_k\}$ is a set of linear operators satisfying

$$\sum_k A_k A_k^\dagger = I.$$

Here I denotes the identity operator acting on the Hilbert space of the system.

Define the Holevo capacity $\chi^*(\Phi)$ of a quantum channel Φ. Suppose Alice has a finite set $\mathcal{M}$ of classical messages that she wants to send to Bob via a memoryless quantum channel Φ. Explain how she can do this, carefully defining all relevant quantities, e.g. the average probability of error, the rate of information transmission, achievable rate and capacity. What is the operational significance of the Holevo capacity?

If the additivity of the Holevo capacity holds, then the classical capacity of a quantum channel cannot be increased by using entangled input states.

Is this statement true or false? Justify your answer.

END OF PAPER