MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 9.00 to 12.00

PAPER 6

REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

Attempt the ***FIRST TWO*** *questions and* ***ANY TWO*** *of the last three questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$. Give examples of two different non-standard Frobenius endomorphisms on G, corresponding to rational structures over the same field $\mathbb{F}_q$; in each case, find a rational maximal torus inside a rational Borel subgroup of G.

(Show that the endomorphisms are indeed distinct. Express the maximal tori and Borel subgroups as conjugates of some fixed subgroups; you do not have to compute explicit matrix entries.)

2 Let $G = \mathrm{SL}_3(\overline{\mathbb{F}}_q)$ with its standard Frobenius endomorphism, and let T be the maximal torus consisting of diagonal matrices in G. Determine, as explicitly as possible, the condition on $\theta \in \mathrm{Irr}(T^F)$ for $R_T^G\theta$ to be irreducible.

3 Let G be a connected reductive group, defined over $\mathbb{F}_q$, with Frobenius endomorphism F. Denote by $\mathcal{B}$ the set of all Borel subgroups of G.

(a) Let G act on $\mathcal{B} \times \mathcal{B}$ by $g(B_1, B_2) = ({}^gB_1, {}^gB_2)$. Show that the orbits in $\mathcal{B} \times \mathcal{B}$ are in bijection with the Weyl group W of G, and that this is equivalent to the Bruhat decomposition.

(b) Let $O(w)$ denote the orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to the element $w \in W$, and let $X(w)$ denote the variety

$$\{B \in \mathcal{B} \mid (B, F(B)) \in O(w)\}.$$

Show that $X(w)$ can be identified with the variety $L^{-1}(BwB)/B$, for a rational Borel subgroup B, and show that there exists a rational maximal torus T' inside a Borel subgroup B' of G, such that there is an isomorphism between $L^{-1}(R_u(B'))/T'^F$ and $L^{-1}(BwB)/B$.

4 State and prove the Mackey formula for Harish-Chandra induction *(you may use any results stated as lemmas in the lectures without proof, but these should be referred to clearly)*.

5 Let G be a connected reductive group, defined over $\mathbb{F}_q$. Let T be a rational maximal torus and $\theta \in \mathrm{Irr}(T^F)$ a character such that $\pm R_T^G\theta$ is irreducible. Show that $\pm R_T^G\theta$ is cuspidal if and only if T is not contained in any proper rational parabolic subgroup of G.

END OF PAPER