UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Monday 2 June 2008 1.30 to 3.30

PAPER 58

QUANTUM INFORMATION, ENTANGLEMENT AND NONLOCALITY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The two following states

$$| \Psi_1 \rangle = \sqrt{p} | 000 \rangle_{ABC} + \sqrt{1-p} | 111 \rangle_{ABC}, \qquad | \Psi_2 \rangle = \frac{1}{\sqrt{2}} (| 000 \rangle_{ABC} + | 111 \rangle_{ABC})$$

are *asymptotically equivalent* under local operations and classical communication (LOCC). That is, there exists an LOCC protocol which has probability $P(n)$ of transforming n copies of $| \Psi_1 \rangle$ into $f(n)$ copies of $| \Psi_2 \rangle$, and then back to n' copies of $| \Psi_1 \rangle$, where $P(n) \to 1$ as $n \to \infty$ and $n' = n[1 - O(1/\sqrt{n})]$.

(i) Calculate the pure state entanglement (measured by the von Neumann entropy) across all bi-partitions and determine the relative entropy of entanglement of all pairwise reduced density matrices for both states. Use your results to determine $f(n)$ to leading order in n.

(ii) Sketch an explicit protocol for the asymptotic transformation of $| \Psi_1 \rangle$ into $| \Psi_2 \rangle$. Verify that the value of $f(n)$ produced by this protocol matches the result obtained in (i).

2 Consider a linear operation $\mathcal{O}$ which, given a quantum state ρ as input, produces as output a classical "outcome" k together with a quantum state ρ_k, with the probability of the k-th outcome being P_k. Let μ be a real function defined on density matrices, and define

$$F(\rho) = \inf \sum_j p_j \mu(\rho_j) \,,$$

where the infimum is taken over all decompositions $\rho = \sum_j p_j \rho_j$ of ρ into a mixture of pure states ρ_j. Prove that if F is non-increasing under the operation $\mathcal{O}$ for pure initial states, then it is also non-increasing if the initial state is mixed, i.e. that $F(\rho) \geqslant \sum_k P_k F(\rho_k)$, for any quantum state ρ with output states $\{\rho_k\}$ and probabilities P_k.

(You may use without proof Bayes' Theorem, i.e. that

$$\text{Prob}(jk) = \text{Prob}(k)\text{Prob}(j|k) = \text{Prob}(j)\text{Prob}(k|j) \,,$$

where $\text{Prob}(jk)$ is the joint probability of events j and k, $\text{Prob}(j)$ is the probability of event j, $\text{Prob}(j|k)$ is the conditional probability of event j given event k, and so on. You may find it helpful to consider a system prepared in a randomly chosen state ρ_j before the operation is applied, and to take $\text{Prob}(jk)$ to be the probability of outcome k and preparation ρ_j, $\text{Prob}(j)$ to be the probability of preparation ρ_j, $\text{Prob}(k)$ to be the probability of outcome k, $\text{Prob}(k|j)$ to be the probability of outcome k given that the operation was performed on preparation ρ_j, and $\text{Prob}(j|k)$ to be the probability that state ρ_j was prepared given that outcome k was obtained.)

3 (i) State and prove the Schmidt decomposition theorem for a pure state $|\psi\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of finite dimensions n_1 and n_2 respectively, with $n_1 \leqslant n_2$.

(ii) Describe the quantum teleportation protocol for transmitting an unknown qubit from A to B, using two bits of classical communication, one maximally entangled shared state of two qubits, and local operations.

4 State EPR's proposed criterion for identifying an element of physical reality. A physical system of N separated particles is described by N qubits in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow_1 \ldots \uparrow_N\rangle - |\downarrow_1 \ldots \downarrow_N\rangle\big) .$$

By considering the action of the operators $\sigma_x \otimes \sigma_y \otimes \sigma_y \ldots \otimes \sigma_y$, $\sigma_y \otimes \sigma_x \otimes \sigma_y \ldots \otimes \sigma_y$, $\ldots$, $\sigma_y \otimes \sigma_y \otimes \sigma_y \ldots \otimes \sigma_x$ and $\sigma_x \otimes \sigma_x \otimes \sigma_x \ldots \otimes \sigma_x$ on $|\psi\rangle$, show that the predictions of quantum theory for the outcomes of local measurements on this state conflict with those implied by the EPR criterion when $N = 4M + 3$, for any non-negative integer M.

END OF PAPER