MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 9.00 to 12.00

PAPER 54

THE STANDARD MODEL

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You may find the following information helpful:

the Pauli matrices may be written

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Weinberg angle θ_W is related to the U(1) and SU(2) gauge couplings g', g of the Standard Model via $\tan\theta_W = g'/g$. Dirac matrices satisfy

$$\gamma_5 \gamma_{\nu_1} \gamma_{\nu_2} \ldots \gamma_{\nu_n} \gamma_5 = (-1)^n \gamma_{\nu_1} \gamma_{\nu_2} \ldots \gamma_{\nu_n},$$

$$\gamma_\nu^\dagger = \gamma_0 \gamma_\nu \gamma_0, \qquad \gamma_5^2 = 1,$$

$$Tr\left[\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma\right] = 4\left(g_{\mu\nu} g_{\rho\sigma} - g_{\nu\rho} g_{\mu\sigma} + g_{\nu\sigma} g_{\mu\rho}\right),$$

$$Tr\left[\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_5\right] = 4\, i\, \epsilon_{\nu\mu\rho\sigma}.$$

Writing Θ as a CPT transformation operator and $\psi(x)$ as a fermionic field, it can be shown that

$$\Theta \psi(x)\, \Theta^\dagger = (\psi^\dagger(-x)\, \gamma_5)^T.$$

The differential cross-section for scattering two approximately massless particles of 4-momenta p_1, p_2 into n massless particles of 4-momenta q_i is, in terms of the matrix element $\mathcal{M}$ of the process,

$$d\sigma = \prod_{i=1}^{n} \left(\frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \right) \frac{|\mathcal{M}|^2}{4p_1.p_2} (2\pi)^4\, \delta^4\, (p_1 + p_2 - \sum_{i=1}^{n} q_i).$$

1 Write down a gauge transformation of the Higgs doublet ϕ, in the Standard Model, specifying its hypercharge. Write down the covariant derivative $D_\mu\phi$ in terms of the SU(2) gauge fields A^a_μ, gauge coupling g and the hypercharge gauge boson B_μ that couples with gauge coupling g'. Going to unitary gauge after spontaneous symmetry breaking (SSB), we may write the Higgs field as

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}.$$

v is the vacuum expectation value of the Higgs Field.

After SSB, we obtain the physical gauge fields

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu\,, \quad A_\mu = \sin\theta_W A^3_\mu + \cos\theta_W B_\mu\,, \quad W_\mu = \frac{1}{\sqrt{2}}(A^1_\mu - iA^2_\mu)\,.$$

In unitary gauge, derive

$$D_\mu\phi = \begin{pmatrix} \dfrac{ig}{2}(v+H)W_\mu \\ \\ \dfrac{1}{\sqrt{2}}\partial_\mu H - \dfrac{ig(v+H)}{2\cos\theta_W\sqrt{2}} Z_\mu \end{pmatrix}.$$

Given the Higgs potential $V(\phi) = \lambda(\phi^\dagger\phi - v^2/2)^2/2$, write down the Standard Model Higgs Lagrangian ignoring interactions with fermions. From this, calculate the masses of the Higgs particle, the Z^0, the photon A and the W bosons at the tree level. Suggest a method of determining $\cos\theta_W$ from empirical measurements of such masses. Draw the Feynman diagrams of all interactions between Higgs particles and physical electroweak gauge bosons, writing the Feynman rule by each diagram. Which interaction was used as the prime Higgs search at the LEP2 collider?

Consider the space-time bi-linear Lagrangian for the W bosons

$$\mathcal{L}_W(x) = -\frac{1}{2}(\partial_\mu W^\dagger_\nu(x) - \partial_\nu W^\dagger_\mu(x))(\partial^\mu W^\nu(x) - \partial^\nu W^\mu(x)) + M_W^2 W^\dagger_\mu(x) W^\mu(x)\,.$$

Derive the momentum space form of the Lagrangian, identifying the operator $O_{\mu\nu}(p)$ explicitly

$$\mathcal{L}_W(p) = -{W_\nu}^\dagger(p) O^{\mu\nu}(p) W_\mu(p)\,.$$

Obtain from $O^{\mu\nu}$ the form of the W propagator in momentum space. What is the low 4-momentum limit of the propagator?

2 Write an essay on the CPT theorem in quantum field theory.

3 The Lagrangian covering the charged current J^μ interactions in the Standard Model is

$$\mathcal{L}_W = -\frac{g}{2\sqrt{2}} \left(J^\mu W_\mu + H.c. \right) ,$$

where W_μ is a W-boson field. The first family leptonic part of the charged current is $J_e^\mu = \nu_e \gamma^\mu (1 - \gamma_5) \bar{e}$. Draw a Feynman diagram and give a Feynman rule for the decay of an on-shell $W^-(p) \to e^-(q_1) \bar{\nu}_e(q_2)$. Assuming that the sum over W polarisation states λ gives

$$\sum_\lambda \epsilon^*_\mu(p, \lambda)\, \epsilon_\nu(p, \lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} ,$$

derive the matrix element squared for the decay (doing appropriate spin sums and averaging):

$$|\bar{M}|^2 = \frac{g^2}{3} \left(q_1 \cdot q_2 + \frac{2(p \cdot q_1)(p \cdot q_2)}{p^2} \right)$$

neglecting fermion masses. Use the formula for the partial width of a 2-body decay

$$\Gamma = \frac{1}{2M_W} \int \frac{d^3 q_1}{(2\pi)^3 2E_{q_1}} \frac{d^3 q_2}{(2\pi)^3 2E_{q_2}} |\bar{M}|^2 \, (2\pi)^4 \, \delta^4(p - q_1 - q_2)$$

in order to show that $\Gamma \propto g^2 M_W$. Find the constant of proportionality. What relationship should there be between this partial width and the partial widths of W^- decaying into the other leptons? Neglecting quark mixing, which quarks can a W^- boson decay into? Estimate the partial widths in each case (assuming massless fermions for the available decays). Draw a table of estimated branching ratios of W^- decays into quarks and leptons, calculating them to the nearest percentage point. What is the total width of the W^-?

After decay of W^- into quarks, briefly describe the subsequent evolution of the quarks.

If we were to include the effects of quark mixing via the CKM matrix V, describe how the partial widths would change. If there are N families of quark, how many degrees of freedom does V contain (show your working)?

4 Consider deep inelastic scattering onto a hadron H of rest-mass M: $e^-(p)H(P) \to e^-(p')X$ via a virtual photon $\gamma(q)$. Denoting the hadron remnant as X, draw a Feynman diagram representing the process and write down a matrix element for it in terms of $q = p - p'$ and the matrix element $\langle X|J_h^\mu|H(P)\rangle$ (J_h^μ being the electromagnetic current of the hadron). You may assume a photon propagator $-ig_{\mu\nu}/Q^2$, where $Q^2 = -q^2$. Thus calculate the differential cross-section in the massless electron approximation

$$\frac{d\sigma}{d^3p'} = \frac{e^4}{(2\pi)^2 8MEE'(Q^2)^2} L_{\mu\nu} W_H^{\mu\nu}(q,P)$$

where E, E' are the electron energies in the rest-frame of H and

$$W_H^{\mu\nu}(q,P) = \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(q+P-p_X)\langle H(P)|J_h^\nu|X\rangle\langle X|J_h^\mu|H(P)\rangle\,,$$

where the spin sum/averaging over hadrons is implicit, thereby identifying $L_{\mu\nu}$ in terms of p and p'.

Provide symmetry arguments to show that we may write

$$W_H^{\mu\sigma}(q,P) = -\left(g^{\mu\sigma} + A\,\frac{q^\mu q^\sigma}{q^2}\right)F_1(x,Q^2) + \left(P^\sigma + B\,\frac{P.q}{q^2}\,q^\sigma\right)\left(P^\mu + C\,\frac{P.q}{q^2}\,q^\mu\right)\frac{F_2(x,Q^2)}{\nu} \qquad (\dagger)$$

where $\nu = P.q$ and $x = Q^2/(2\nu)$, determining the constants A, B and C in the process.

$W_H^{\mu\nu}(q,P)$ is approximated by viewing inclusive scattering as incoherent elastic scattering from point-like constituents ("partons") of momentum fraction ξ of the initial hadron's 4-momentum. The probability distribution function for a quark of flavour f having such a fraction is defined to be $q_f(\xi)$ (for anti-quarks we write $\bar{q}_f(\xi)$). Thus

$$W_H^{\mu\nu}(q,P) = \int_0^1 d\xi \sum_f Q_f^2 \tilde{W}^{\mu\nu}(\xi,q,P)(q_f(\xi)+\bar{q}_f(\xi))\,,$$

where $\tilde{W}^{\mu\nu}(\xi,q,P)$ describes the scattering of $\gamma(q)$ with a parton of 4-momentum ξP into a parton of 4-momentum k, i.e.

$$\tilde{W}^{\mu\nu}(\xi,q,P) = \frac{1}{4\pi\xi}\int \frac{d^3k}{(2\pi)^3 2E_k}(2\pi)^4\delta^4(q-k+\xi P)\langle \xi P|J_h^\nu|k\rangle\langle k|J_h^\mu|\xi P\rangle\,.$$

Show that the only terms to contribute to the scattering (neglecting quark and lepton masses) are

$$\tilde{W}^{\mu\nu}(\xi,q,P) = \frac{\delta(E_q - E_k + \xi E_p)}{E_k}\left[\xi P^\mu P^\nu - \frac{q.P}{2}g^{\mu\nu}\right].$$

Next, show that

$$\delta(E_q + \xi E_p - E_k)/E_k = 2\delta(E_k^2 - (\xi E_p + E_q)^2) = \delta(x-\xi)/\nu$$

and hence, by comparison with Eq. (†), show that the Callan-Gross relation holds, i.e.

$$2xF_1(x,Q^2) = F_2(x,Q^2)\,.$$

END OF PAPER