UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 1.30 to 4.30

PAPER 52

STRING THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Minor errors in numerical factors will not be heavily penalized.

The covariant world-sheet action for the bosonic string in flat space-time is

$$I = \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \, \sqrt{-\det\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X \ .$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the local symmetries of the bosonic string action.

Explain how the physical states of the closed bosonic string are defined in the parametrization of the world-sheet in which the two-dimensional metric is flat (i.e., in the 'conformal gauge'). Determine the residual symmetry and its generators in this gauge.

'Bosonic string perturbation theory can be expressed as a theory of two dimensional quantum gravity.' Explain this statement.

What is the evidence that closed bosonic string theory describes the propagation of gravity in the d-dimensional space-time through which the string is moving?

2 Explain, without mathematical details, how the physical states of the open bosonic string with Neumann boundary conditions are defined in the light-cone gauge, starting from the classical world-sheet action.

Show that the number of states at level p of the open string with Neumann boundary conditions is given by

$$d_p = \frac{1}{2\pi i} \oint F(w)\, w^{-p}\, dw\,,$$

where $F(w) = \prod_{n=1}^{\infty}(1 - w^n)^{-24}$ and the integration contour is a circle of radius $\ll 1$ around $w = 0$.

Show that for $p \to \infty$ and $w = 1 - \epsilon$ ($\epsilon \ll 1$),

$$\log F(w) = \frac{4\pi^2}{\epsilon} + C(\epsilon),$$

where you do not need to derive the subleading term, $C(\epsilon) = -6 \log \epsilon +$ constant.

By deforming the contour through a saddle point near $w = 1$, or otherwise, show that at large p the degeneracy of states is well approximated by

$$d_p \sim p^{-27/4}\, e^{4\pi\sqrt{p}}$$

(you need not prove that the contribution from the contour far from $w = 1$ can be neglected).

What does this asymptotic spectrum of states suggest about the behaviour of string theory at high temperature?

3 The action for a free scalar field, $\phi(z)$, in two-dimensional euclidean space (z spans the complex plane) is $S[\phi] = \frac{1}{2}\int \partial_\alpha\phi\partial^\alpha\phi\, d^2z$. Show that the functional integral for ϕ coupling to a source, $J(z)$, can be written in the form

$$\int D\phi e^{-S[\phi]-\int J(z)\,\phi\, d^2z} = \mathcal{N}\, e^{-\frac{1}{2}\int J(z')\, G(z',z'')\, J(z'')\, dz' dz''} .$$

Give a formal expression for the J-independent prefactor, $\mathcal{N}$, which you *need not* evaluate. What equation does $G(z', z'')$ satisfy and what is its solution?

The tree amplitude for the scattering of four tachyonic ground states of the closed bosonic string is given by the euclidean functional integral

$$A_4 = \int DX \prod_{r=1}^{4} (\int d^2z_r\, V_0(k_r, z_r))\, e^{-I[X]} .$$

In this expression $\int DX$ indicates a functional integral over the embeddings, X^μ, $I[X]$ is the euclidean string action in a parametrization in which the world-sheet is flat and $V_0(k_r, z_r) = e^{ik_r\cdot X(z_r)}$ is the vertex operator describing the coupling of a tachyonic ground state of momentum k_r^μ to the world-sheet at position z_r. Show that the functional integral can be reduced to the form

$$A_4 = \mathcal{N} \int \prod_{q=1}^{4} d^2z_q \prod_{r<s} |z_r - z_s|^{\alpha' k_r\cdot k_s} ,$$

where $\mathcal{N}$ is a k_r-independent constant.

Show that if the mass, μ, of the ground states has the value given by $-k_r^2 = \mu^2 = -4/\alpha'$ the integrand of A_4 is invariant under the $SL(2, C)$ transformations

$$z_r \to \frac{az_r + b}{cz_r + d}$$

(where a, b, c and d are complex and $ad - bc = 1$).

The result of evaluating the z_r integrals (which you need not attempt) is

$$A_4 = C\, \frac{\Gamma\left(-1 - \frac{\alpha' s}{4}\right)\Gamma\left(-1 - \frac{\alpha' t}{4}\right)\Gamma\left(-1 - \frac{\alpha' u}{4}\right)}{\Gamma\left(2 + \frac{\alpha' s}{4}\right)\Gamma\left(2 + \frac{\alpha' t}{4}\right)\Gamma\left(2 + \frac{\alpha' u}{4}\right)} ,$$

where C is a constant, the Mandelstam invariants are defined by $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$, $u = -(k_1 + k_3)^2$, and the gamma function, $\Gamma(r)$, has simple poles only at $r = 0, -1, -2, \ldots$.

What is the physical interpretation of the poles in s, t and u in this amplitude?

Use the expression for A_4 to infer the value of the coupling between three tachyons (you will need to show that $s + t + u = 4\mu^2$).

4 If ψ^1_a and ψ^2_a are two distinct fermionic Majorana spinor fields ($a = 1, 2$ is a spinor index that is suppressed in the following), prove that

$$\bar\psi^1\,\psi^2 = \bar\psi^2\,\psi^1\,, \qquad \bar\psi^1\,\rho^\alpha\,\psi^2 = -\bar\psi^2\,\rho^\alpha\,\psi^1\,,$$

where the matrices ρ^α are two-dimensional Dirac matrices and $\bar\psi = \psi^T\rho^0$.

A fermionic extension of the bosonic string in Minkowski space-time is obtained by including world-sheet Majorana fermionic fields, $\psi^\mu_a(\sigma,\tau)$ ($\mu = 0, 1, \ldots, 9$ is a space-time vector index), with the Dirac action,

$$I^\psi = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\,\bar\psi\cdot\rho^\alpha\partial_\alpha\psi\,,$$

Show that the sum of the bosonic and fermionic actions is invariant under the two-dimensional fermionic symmetry (supersymmetry) associated with the transformations

$$\delta_\epsilon X^\mu = -\bar\epsilon\,\psi^\mu\,, \qquad \delta_\epsilon\psi^\mu = \rho^\alpha\,\partial_\alpha X^\mu\,\epsilon\,,$$

where ϵ^a is the constant anticommuting Majorana spinor parameter of the transformation.

The commutator of two supersymmetry transformations is defined by $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_1}\,\delta_{\epsilon_2} - \delta_{\epsilon_2}\,\delta_{\epsilon_1}$, where ϵ_1 and ϵ_2 are two independent spinor parameters. Show that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,X^\mu = \ell^\alpha\,\partial_\alpha X^\mu\,, \qquad [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\psi^\mu = \ell^\alpha\,\partial_\alpha\psi^\mu\,,$$

where the vector ℓ^α is a specific translation of the world-sheet parameters, σ and τ, which should be determined. [In evaluating the second commutator you should pay attention to the labelling of the suppressed spinor indices, perhaps by using an explicit representation of ρ^0 and ρ^1, and you should drop terms that vanish on using the ψ^μ equation of motion on the right-hand side of the second equation.]

Comment (without details) on how this symmetry leads to a fermionic extension of the Virasoro algebra.

[*You may use the representation of the Dirac matrices*

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad]$$

END OF PAPER