UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 9.00 to 12.00

PAPER 50

STATISTICAL FIELD THEORY AND APPLICATIONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain briefly what is meant by a phase diagram. Give an example of a three-dimensional phase diagram which contains a tricritical point and describe the nature of the phase transitions which can occur.

Give an account of the Landau-Ginsberg (LG) theory of phase transitions illustrated using scalar field theory and including a discussion of the following topics:

(a) The idea of an *order parameter*;

(b) The definition of the *correlation length* ξ and its rôle in LG theory;

(c) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(d) The concept of universality and which properties are, and are not, universal at a critical point;

(e) The idea of *critical exponents* and how they may be derived;

(f) The features of a tricritical point and how it occurs in LG theory.

You should clarify your account with diagrams which should be appropriately labelled.

The critical indices β, γ, δ in the Ising model are defined by

$$M \sim (-t)^{\beta} \quad (t < 0, h = 0), \qquad \chi \sim t^{-\gamma} \quad (t > 0, h = 0)\ , \qquad M \sim h^{1/\delta} \quad (t = 0, h > 0),$$

where M is the magnetization, χ is the susceptibility, $t = (T - T_C)/T_C$ and h is the applied magnetic field. Calculate β, γ, δ for both an ordinary critical point and a tricritical point in LG theory.

Explain how the Ginsberg criterion shows that LG theory fails to predict the correct critical exponents for a general critical point if the dimension D satisfies $D \leq D_c$. Derive the expression for D_c in the general case and show that

$$D_c = 4 \quad \text{for an ordinary critical point,} \qquad D_c = 3 \quad \text{for a tricritical point.}$$

2 The Ising model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u}, \sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$ and $\sigma_r \in \{1, -1\}$. In particular, $\mathcal{H}$ contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u}, \sigma) - \beta NC) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour for $r \gg \xi$.

State how the magnetization M and the magnetic susceptibility χ can be determined from the free energy F, and derive the relation which expresses χ in terms of $G(\mathbf{r})$.

Explain how a renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$. Why does $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$ predict the same large-scale properties of the system as does $\mathcal{Z}(\mathbf{u}, C, N)$?

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$ and explain how it may be expressed in terms of a singular part $f(\mathbf{u})$ which obeys the RG equation

$$f(\mathbf{u}_0) \;=\; b^{-pD} f(\mathbf{u}_p) + \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j) \;, \qquad (*)$$

where the rôle of the function $g(\mathbf{u})$ should be explained.

Explain the ideas of a fixed point, a critical surface, relevant and irrelevant operators, and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived, and explain under what conditions the second term (the inhomogeneous term) on the right-hand-side of $(*)$ may be safely neglected.

In the case that there are two relevant couplings $t = (T - T_C)/T_C$ and h, derive the scaling hypothesis for the free energy $F_\pm(r, h, C_0)$:

$$F_\pm \;=\; |t|^{D/\lambda_t}\left(f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) + I_\pm \right) \;,$$

What is the significance of the subscript label $\pm$ on these functions?

Briefly explain why the amplitude ratio $F_+(t, h = 0)/F_-(t, h = 0)$ is expected to be universal.

The following critical exponents are defined:

$$\begin{aligned} \xi &\sim |t|^{-\nu} \quad && h = 0 \\ C_V &\sim |t|^{-\alpha} \quad && h = 0 \quad \text{(the specific heat)} \\ M &\sim |t|^{\beta} \quad && h = 0, \;\; T < T_C \end{aligned}$$

Under suitable assumptions derive the relation

$$\alpha \;=\; 2 - D\nu \; .$$

where D is the dimension of space.

The Gaussian model in D dimensions ($D \leq 4$) for a real scalar field is defined by the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\left(\kappa^{-1}(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})\right) + h\phi(\mathbf{x}) \; ,$$

where κ, m and h are coupling constants.

Derive an expression for the correlation length ξ in terms of the coupling constants.

By defining a suitable thinning transformation show that the critical exponents α and β are given by

$$\alpha = (4 - D)/2 \; , \qquad \beta = (D-2)/4 \; .$$

3 The partition function of the XY model can be written

$$Z \;=\; \int \prod_{\mathbf{x}'} d\theta(\mathbf{x}') \exp\left[-\beta \sum_{\mathbf{x},\mu} \Big(1 - \cos\left[\theta(\mathbf{x} + \hat{\mu}) - \theta(\mathbf{x})\right] \Big) \right] .$$

(a) Use the fact that, for large β

$$\exp(-\beta(1 - \cos\psi)) \;\simeq\; \frac{1}{\sqrt{2\pi\beta}} \sum_{n=-\infty}^{\infty} e^{in\psi}\, e^{-n^2/2\beta}$$

to write down the partition function for the Villain model.

(b) Take this model as valid for all β and perform a duality transformation which separates the partition function into independent contributions from spin waves and vortices. You will use Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} g(n) \;=\; \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\, g(\phi)\, e^{2\pi i m\phi} \; .$$

(c) Describe the interactions between vortices.

(d) Derive a constraint on the net vorticity.

4 A scalar field theory in $D = 4 - \epsilon$ dimensions near a critical point is described by the Hamiltonian density

$$\mathcal{H}(\phi) \;=\; \frac{1}{2}|\nabla\phi(\mathbf{x})|^2 \;+\; \frac{1}{2}m^2(\Lambda, T)\phi^2(\mathbf{x}) \;+\; \frac{1}{4!}g(\Lambda, T)\phi^4(\mathbf{x})$$

where T is the temperature and Λ is the ultraviolet cutoff.

(a) By requiring that the properties of the theory be independent of the choice of Λ, show that the dimensionless couplings $(u^2, \lambda) = (m^2\Lambda^{-2}, g\Lambda^{-\epsilon})$ obey the renormalization group flow equations, correct to lowest order in ϵ, of the form

$$\frac{du^2}{db} \;=\; 2u^2 \;+\; \frac{\Omega_D}{2(2\pi)^D}\frac{\lambda}{1+u^2}$$

and

$$\frac{d\lambda}{db} \;=\; \epsilon\lambda \;-\; \frac{3}{2}\frac{\Omega_D}{(2\pi)^D}\frac{\lambda^2}{(1+u^2)^2}$$

where Ω_D is the area of a unit sphere in D dimensions and $b = \log(\Lambda_0/\Lambda)$, with Λ_0 the initial cutoff.

(b) Treating ϵ as a small positive parameter, show that these equations have an infrared attractive fixed point at

$$u_*^2 \;=\; -\frac{\epsilon}{6} \qquad \lambda_* \;=\; \frac{16\pi^2\epsilon}{3}\,.$$

(c) Explain why a fixed point of this nature at $\lambda_* \neq 0$ means that the Landau-Ginsberg approach is invalid.

[You may quote the Feynman rules of perturbation theory without derivation.]

END OF PAPER