UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 9.00 to 12.00

PAPER 49

SYMMETRY AND PARTICLE PHYSICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{L}$ be a Lie algebra. Define the Killing form κ of $\mathcal{L}$, and prove that

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

for $X,Y,Z \in \mathcal{L}$. Derive an expression for the components of κ in terms of the structure constants of $\mathcal{L}$ with respect to an arbitrary basis of generators of $\mathcal{L}$. What does it mean for $\mathcal{L}$ to be semi-simple or compact?

(i) Suppose that $\mathcal{L}$ is a three-dimensional Lie algebra with generators K_1, K_2, J satisfying the commutation relations

$$[K_1,\ K_2] = J, \qquad [J,\ K_1] = \lambda K_2, \qquad [J,\ K_2] = -\lambda K_1$$

for $\lambda \in \mathbb{R}$. Compute the Killing form of this Lie algebra, and determine, for all possible values of λ, whether $\mathcal{L}$ is semi-simple and whether $\mathcal{L}$ is compact.

(ii) Let G be the three-dimensional matrix Lie group defined by

$$G = \left\{ \begin{pmatrix} \cosh\rho & \sinh\rho & x \\ \sinh\rho & \cosh\rho & y \\ 0 & 0 & 1 \end{pmatrix} \quad : \quad \rho, x, y \in \mathbb{R} \right\}$$

Taking ρ, x, y as co-ordinates on G, compute a basis of left-invariant vector fields in $T(G)$, and evaluate the Haar measure.

2 Let d be a finite-dimensional anti-hermitian representation of $\mathcal{L}(SU(3))$ acting on a complex vector space V. Show that there are three complexified $\mathcal{L}(SU(2))$ subalgebras associated with this representation. Describe briefly, with illustrations, how these subalgebras can be used to classify such representations with weight diagrams, stating (without proof) the symmetries and the multiplicity rules for the weight diagrams.

Suppose that the highest weight is (p, q), and let

$$X = (H_1)^2 + (H_2)^2 + \sum_{i=1}^{3} \left(E_+^i E_-^i + E_-^i E_+^i \right) .$$

Prove that

$$X = \left(p + \sqrt{3}q + p^2 + q^2\right)\mathcal{I}_V$$

where $\mathcal{I}_V$ denotes the identity operator on V.

What are the possible values of X when evaluated on $SU(3)_{flavour}$ multiplets of light hadrons?

[*You may assume the standard non-vanishing commutation relations*

$$[H_1, E_\pm^1] = \pm E_\pm^1 , \qquad [H_1, E_\pm^2] = \mp\frac{1}{2} E_\pm^2 , \qquad [H_1, E_\pm^3] = \pm\frac{1}{2} E_\pm^3 ,$$

and

$$[H_2, E_\pm^1] = 0 , \qquad [H_2, E_\pm^2] = \pm\frac{\sqrt{3}}{2} E_\pm^2 , \qquad [H_2, E_\pm^3] = \pm\frac{\sqrt{3}}{2} E_\pm^3 ,$$

and

$$[E_+^1, E_-^1] = H_1 , \quad [E_+^2, E_-^2] = \frac{\sqrt{3}}{2} H_2 - \frac{1}{2} H_1 , \quad [E_+^3, E_-^3] = \frac{\sqrt{3}}{2} H_2 + \frac{1}{2} H_1 ,$$

and

$$\begin{aligned}
[E_+^1, E_+^2] &= \frac{1}{\sqrt{2}} E_+^3 , \qquad [E_-^1, E_-^2] = -\frac{1}{\sqrt{2}} E_-^3 , \\
[E_+^1, E_-^3] &= -\frac{1}{\sqrt{2}} E_-^2 , \quad [E_-^1, E_+^3] = \frac{1}{\sqrt{2}} E_+^2 , \\
[E_+^2, E_-^3] &= \frac{1}{\sqrt{2}} E_-^1 , \qquad [E_-^2, E_+^3] = -\frac{1}{\sqrt{2}} E_+^1 ,
\end{aligned}$$

where $iH_1, iH_2, i(E_+^m + E_-^m), E_+^m - E_-^m$ *are the antihermitian* $\mathcal{L}(SU(3))$ *generators in the representation* d *and* $H_1 = d(h_1)$, $H_2 = d(h_2)$ *with*

$$h_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad h_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} .]$$

3 Let $\mathcal{L}(G)$ be the matrix Lie algebra of a matrix Lie group G, and let A_μ be a $\mathcal{L}(G)$-valued gauge potential with Yang-Mills field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where $\mu, \nu = 0, 1, 2, 3$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. By considering the fundamental covariant derivative, determine how A_μ transforms under gauge transformations. Also determine how $F_{\mu\nu}$ transforms under gauge transformations.

Suppose that $\mathcal{L}(G)$ is semi-simple with Killing form κ. Consider the Lagrangian densities L_1 and L_2 defined by

$$L_1 = \big(\kappa(F_{\mu\nu}, F^{\mu\nu})\big)^p \quad , \qquad\qquad L_2 = \kappa(D^\mu F^{\nu\rho}, D_\mu F_{\nu\rho})$$

where in L_1, p is a fixed positive integer and in L_2, D_μ denotes the adjoint covariant derivative.

Prove that both L_1 and L_2 are gauge-invariant, and evaluate the gauge field equations associated with both L_1 and L_2.

[*In this question, you may assume* $\kappa(X, [Y, Z]) = \kappa([X, Y], Z)$ *for* $X, Y, Z \in \mathcal{L}(G)$, *and that if* $X \in \mathcal{L}(G)$ *and* $g \in G$ *then* $gXg^{-1} \in \mathcal{L}(G)$.]

4 Give an account of the theory of unitary representations of the Poincaré group.

You should prove how the Pauli-Lubanski vector W^μ of a representation of the Poincaré algebra, and the "little group", can be used to classify the representation in terms of timelike or null 4-momenta of physical interest, giving a detailed description of the timelike case.

[*For the purposes of this question, you may assume without proof the Poincaré algebra:*

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= i\big(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\big) \\
[P^\mu, M^{\rho\sigma}] &= i\eta^{\mu\rho}P^\sigma - i\eta^{\mu\sigma}P^\rho \\
[P^\mu, P^\nu] &= 0
\end{aligned}$$

and the following commutation relations involving W^μ

$$[W^\mu, d(P^\nu)] = 0, \qquad [W^\mu, d(M^{\rho\sigma})] = i\eta^{\mu\rho}W^\sigma - i\eta^{\mu\sigma}W^\rho \ .$$

You may also assume that a unitary representation $\mathcal{D}$ *of the Poincaré group acting on a vector space* V *gives rise to a representation* d *of the Poincaré algebra related via*

$$\mathcal{D}(e^{-\frac{i}{2}(b_\mu P^\mu + \omega_{\mu\nu}M^{\mu\nu})}) = e^{-\frac{i}{2}(b_\mu d(P^\mu) + \omega_{\mu\nu} d(M^{\mu\nu}))}$$

for real b_μ *and skew-symmetric real* $\omega_{\mu\nu}$, *and that* $d(P^\mu)$ *and* $d(M^{\rho\sigma})$ *are hermitian.*]

END OF PAPER