UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 9.00 to 11.00

PAPER 40

OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) An investor can invest in a riskless bank account, with constant rate of return r, or in a log-Brownian asset with volatility σ and mean growth rate μ, so that his wealth w_t at time t satisfies

$$dw_t = rw_t dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt,$$

where c_t is his rate of consumption at time t, W is a standard Brownian motion, and θ_t is the value at time t of the investor's holding of the risky asset. Suppose his objective is to maximise

$$E\int_0^\infty e^{-\rho t}U(c_t)\, dt,$$

subject to $w_t \geqslant 0$ for all t, where $U : (0, \infty) \to \mathbb{R}$ is C^2, strictly increasing and strictly concave, $U'(0) = \infty,\ U'(\infty) = 0$. Derive the form of his optimal consumption process, and characterise the corresponding θ as explicitly as you can.

[Standard results from stochastic integration theory may be used provided they are clearly stated.]

(b) Assuming now that $U(x) = x^{1-R}/(1 - R)$ for some positive $R \neq 1$, find the value function, optimal portfolio θ and optimal consumption rate c_t. When is the problem well posed? What happens if it is not?

2 A continuous-time model of an economy contains a single productive asset, whose dividend process $(\delta_t)_{t\geqslant 0}$ evolves as

$$d\delta_t = \delta_t(\sigma dW_t + \mu dt),$$

where W is a standard Brownian motion. Agent $i \in \{1, \ldots, J\}$ has preferences over consumption streams $(c_t^i)_{t\geqslant 0}$ given by

$$E\int_0^\infty e^{-\rho_i t}U_i(p_t^i)\, dt,$$

where

$$p_t^i = \frac{c_t^i}{\sum_j c_t^j}$$

and $U_i : (0, \infty) \to \mathbb{R}$ is C^2, strictly increasing and strictly concave, $U'(0) = \infty,\ U'(\infty) = 0$. Agent i initially holds a fraction π_0^i of the productive asset.

By considering marginal pricing of future consumption relative to present consumption, or otherwise, derive the equilibrium for this economy as explicitly as you can. Show that in equilibrium the process $(p_t)_{t\geqslant 0}$ is non-random. Show that if all the agents have same ρ_i and U_i then in equilibrium the proportion of the output of the productive asset which agent i consumes is constant and equal to π_0^i.

3 An agent works until a time τ of his choosing, at which time he retires. While working, he receives a constant income stream of ε, which incurs a disutility $\lambda > 0$. He invests his wealth in a riskless bank account bearing interest rate r, and in a risky stock with constant volatility σ and rate of growth μ. His wealth therefore evolves as

$$dw_t = rw_t dt + \theta(\sigma dW_t + (\mu - r)dt) - c_t dt + \varepsilon I_{\{t \leqslant \tau\}} dt$$

(where W is a standard Brownian motion, and θ_t is the time-t value of his holding of the stock) and he seeks to maximise

$$E \int_0^\infty e^{-\rho t}(U(c_t) - \lambda I_{\{t \leqslant \tau\}})\, dt.$$

Assume that $U'(x) = x^{-R}$ for some positive constant $R \neq 1$. Show that the critical level at which he retires is $\gamma_M^{-1}(\varepsilon/\lambda)^{1/R}$. By introducing the dual variable $z = V'(w)$, solve his problem as completely as you can.

4 An agent may invest in a riskless bank account with constant interest rate $r \geqslant 0$ and in a risky stock whose price S_t at time t evolves as

$$dS_t = S_t(\sigma(\xi_t)dW_t + \mu(\xi_t)dt),$$

where ξ is an irreducible Markov chain on the finite set I with jump intensity matrix Q, independent of the standard Brownian motion W.

The agent's objective is to maximise

$$E \int_0^\infty e^{-\rho t} U(c_t) dt$$

where c_t is the consumption rate at time t, under the constraint that the wealth at all times should be non-negative. Write down the dynamics for the agent's wealth w_t at time t, and derive the Hamilton-Jacobi-Bellman (HJB) equation to be satisfied by the value function.

Assuming that $U'(x) = x^{-R}$ for some positive $R \neq 1$, find a simplification of the HJB equation. Briefly outline how you would go about solving this equation numerically.

Briefly explain what further steps are required to prove that the candidate for the value function which you have found using the HJB equations is actually optimal.

END OF PAPER