UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 9.00 to 12.00

PAPER 39

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

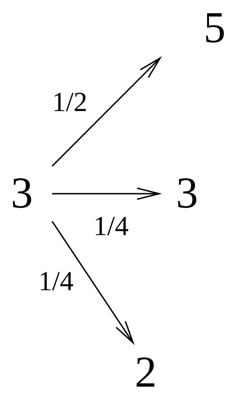
1 Consider a one-period two-asset market model $(B_t, S_t)_{t\in\{0,1\}}$, where $B_t > 0$.

(i) What is an arbitrage strategy?

(ii) What is an equivalent martingale measure?

(iii) Show that there is no arbitrage if there exists an equivalent martingale measure.

Now consider the following model. Suppose that asset 0 is cash, so that $B_0 = B_1 = 1$. Asset 1 is a stock with price modelled by



(The diagram should be read $S_0 = 3$ and $\mathbb{P}(S_1 = 5) = 1/2$, etc., where $\mathbb{P}$ is the objective probability measure.)

Now introduce a call option with strike 4 maturing at time 1.

(iv) Find an arbitrage strategy in the case that the time-0 price of this option is $1/2$.

(v) What are the possible time-0 prices of this option such that the market has no arbitrage?

2 Let $(B_t, S_t)_{t\in\mathbb{Z}_+}$ be a model of an arbitrage-free financial market with two assets, where $B_t > 0$.

(i) What does it mean to say that the market is complete?

(ii) Show that if the market is complete, the equivalent martingale measure is unique.

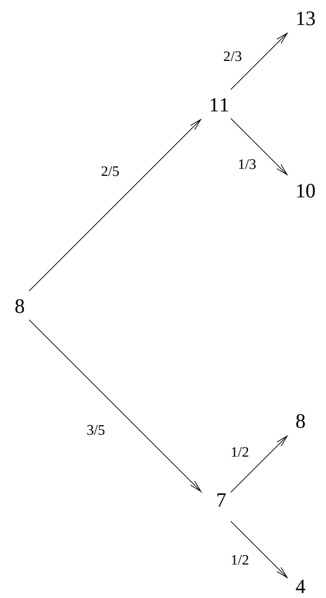
Now suppose the market is complete, and that $B_{t+1} \geq B_t$ for all $t \in \mathbb{Z}_+$. Let $C(T, K)$ be the time-0 price of a European call option with strike K and maturity T.

(iii) Show that $T \mapsto C(T, K)$ is increasing, and that $K \mapsto C(T, K)$ is decreasing and convex.

3 (i) What does it mean to say the process $(U_t)_{t\in\mathbb{Z}_+}$ is a supermartingale?

(ii) Let $(U_t)_{t\in\mathbb{Z}_+}$ be a supermartingale and τ a stopping time. Show that the process $(U_{t\wedge\tau})_{t\in\mathbb{Z}_+}$ is a supermartingale.

Let $(\xi_t)_{t\in\{0,1,2\}}$ be as follows



where the above diagram should be read as

$$\mathbb{P}(\xi_1 = 11) = 2/5, \quad \mathbb{P}(\xi_2 = 13|\xi_1 = 11) = 2/3, \text{ etc.}$$

(iii) Show that $\mathbb{E}(\xi_\tau) \leq 9$ for every stopping time τ.

(iv) Find a stopping time τ_0 such that $\mathbb{E}(\xi_{\tau_0}) = 9$.

4 Consider a Black–Scholes market with two assets whose dynamics are given by

$$dB_t = B_t \; r \; dt$$
$$dS_t = S_t(\mu \; dt + \sigma dW_t).$$

(i) Show the density process $(Z_t)_{t \in \mathbb{R}_+}$ of the equivalent martingale measure is of the form

$$Z_t = e^{-\lambda^2 t/2 - \lambda W_t}$$

for a parameter λ to be determined.

Now introduce a European claim maturity at time T with payout $\xi = \max\{K, S_T\}$.

(ii) Find a no-arbitrage price process $(\xi_t)_{t \in [0,T]}$ such that $\xi_T = \xi$.

(iii) Find the replicating strategy $(\pi_t)_{t \in [0,T]}$ for the claim.

5 Consider a Markovian market model with two assets whose risk-neutral dynamics are given by

$$dB_t = B_t \; r \; dt$$
$$dS_t = S_t(r \; dt + \sigma(S_t) d\hat{W}_t)$$

for given a continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ and a Brownian motion $(\hat{W}_t)_{t \in \mathbb{R}_+}$.

(i) Introduce a European contingent claim with maturity T and payout $\xi = g(S_T)$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and continuous. Show that there is no arbitrage in the augmented market $(B_t, S_t, \xi_t)_{t \in [0,T]}$ if

$$\xi_t = V(t, S_t)$$

and $V : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satifies the partial differential equation

$$\frac{\partial V}{\partial t}(t,S) + rS\frac{\partial V}{\partial S}(t,S) + \frac{1}{2}\sigma(S)^2 S^2 \frac{\partial^2 V}{\partial S^2}(t,S) = rV(t,S)$$
$$V(T,S) = g(S).$$

(ii) How can the replicating portfolio π_t be calculated in terms of the function V?

(iii) Now specialize to the case where $r = 0$, $\sigma(S) = S^{-1/2}$, and $g(S) = e^S$, and $T = 1$. By making the substitution $V(t,S) = e^{A(t)S + B(t)}$, find the time-$t$ option price in this model.

6 Let $(r_t)_{t\in\mathbb{R}_+}$ be an interest rate process modelled by

$$r_t = g(t) + \sigma \hat{W}_t$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a Brownian motion for the unique equivalent martingale measure, the function $g : \mathbb{R}_+ \to \mathbb{R}$ is given, and $\sigma > 0$ is constant.

(i) Compute the bond prices $P_t(T)$ and the forward rates $f_t(T)$ for this model.

(ii) Show that one can choose the function g in such a way as to exactly match any time-0 forward rate curve $f_0(\cdot)$.

END OF PAPER