UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 4.30

PAPER 38

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let P be the linear programming problem: maximize $\{c^\top x \,:\, Ax \leqslant b,\ x \geqslant 0\}$, where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is $m \times n$. What is its dual, D?

Explain why the following are true.

(a) If P is feasible then D is bounded.

(b) If P is feasible and bounded then D is feasible and bounded.

Suppose that the polytope $\Pi = \{x \,:\, Ax \leqslant b,\ x \geqslant 0\}$ is empty. Show that there exists some $\lambda \geqslant 0$ such that $\lambda^\top A \geqslant 0^\top$ and $\lambda^\top b = -1$.

Let $\Pi(\epsilon) = \{x \,:\, Ax \leqslant b + \epsilon e,\ x \geqslant 0\}$, where $e^\top = (1, \ldots, 1)$ and $\epsilon \geqslant 0$. Given that $\Pi = \Pi(0)$ is empty, and λ is as above, show that $\Pi(\epsilon)$ is empty for all ϵ such that $0 \leqslant \epsilon < 1/\sum_i \lambda_i$.

Explain the relevance of this result to the theory of the ellipsoid algorithm.

2 The Klee-Minty polytope in $\mathbb{R}^3$ is the intersection of the six halfspaces on which $x = (x_1, x_2, x_3)$ satisfies the following constraints, for given ϵ, $0 < \epsilon < 1/2$:

$$\begin{aligned} x_1 &\geqslant 0\,, \\ x_1 &\leqslant 1\,, \\ x_2 &\geqslant \epsilon x_1\,, \\ x_2 &\leqslant 1 - \epsilon x_1\,, \\ x_3 &\geqslant \epsilon x_2\,, \\ x_3 &\leqslant 1 - \epsilon x_2\,. \end{aligned}$$

This polytope, P, has six facets, which are respectively indexed as $1, 2, \ldots, 6$, as they lie on a boundary of each of the above six halfspaces. The vertex at the intersection of the first, third and fifth facets is $v_0 = (0, 0, 0)$. Bland's pivot rule says that at each successive step of the simplex algorithm the solution should move from the current vertex of the feasible set to an adjacent vertex; the objective function value should increase, and if there is more than one adjacent vertex at which that value increases then we should pick the one that we move to along the edge that is leaving the facet of smallest index. It is desired to maximize x_3 over P. Show, using a picture, the steps taken by the simplex algorithm under Bland's rule, starting from v_0.

Discuss the worst-case running time of the simplex algorithm under Bland's rule.

Show that there are examples of linear programs, in decision variables $x \in \mathbb{R}^n$, and with $2n$ constraints, in which it takes at least n pivots to move from an initial solution to the optimum, no matter how the pivots are chosen.

An alternative to Bland's pivot rule is Dantzig's rule, in which we are to move to an adjacent vertex along an edge for which the rate of increase in the objective function is greatest. Describe how to modify P to show that there is a similar example for which Dantzig's rule is inefficient.

3 Describe the minimum cost flow problem.

Explain how one can use the Lagrangian sufficiency theorem to identify an optimal solution by means of node numbers.

A network has nodes $N = \{1, 2, 3, 4\}$ and directed arcs $A = \{(1,2), (2,3), (3,4), (4,1)\}$. Nodes 1, 3 and 4 are sources, for flow amounts 1 each. Node 2 is a sink, for flow of amount 3. The minimum permitted flows on arcs $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$ are 3, 2, 2, 0, respectively; the maximum permitted flows are 8, 5, 5, 8, respectively. The costs per unit flow on these arcs are c_{12}, c_{23}, c_{34}, c_{41}, respectively. Show that there is a feasible solution in which arc $(2,3)$ carries flow of 2.

Derive a condition in terms of c_{12}, c_{23}, c_{34}, c_{41} under which this is the minimum cost flow.

Use the network simplex algorithm to find the minimum cost flow for all possible values of the four cost variables $\{c_{ij}\}$, $c_{ij} \in (-\infty, \infty)$.

Determine the numbers of (i) basic solutions, and (ii) basic feasible solutions to this problem.

4 Let $G = (V, E)$ be an undirected graph. Edge e has weight w_e and the edge weights are distinct, say $w_1 < \cdots < w_{|E|}$. Let S be a nonempty proper subset of V, and let edge $e = (u, v)$ be a edge of least weight that has one end in S and the other end in $V \setminus S$. Prove that every minimum spanning tree must contain the edge e.

Use the above result to prove that a minimum spanning tree can be found both by Prim's algorithm (which you should state), and by Kruskal's algorithm (in which we build a spanning tree by successively considering edges in order of increasing edge weight, inserting an edge if this does not create a cycle).

Is the minimum spanning tree unique?

Let C be a cycle in G, and let the edge $e = (v, w)$ be the edge in C of maximum weight. Prove that no minimum spanning tree can contain e. Use this to prove that the minimum spanning tree problem can also be solved by a 'reverse Kruskal's algorithm', in which we start with the full graph (V, E) and then successively consider edges in order of decreasing weight, deleting an edge if this does not disconnect the graph.

5 Describe how to formulate the decision travelling salesman problem (TSP) as a 0–1 integer linear programming problem.

Carefully explain what it means to say that decision TSP is $\mathcal{NP}$-complete.

In 'decision Max-TSP' the aim is to decide where there is a tour of length greater than some given L. Given that TSP is $\mathcal{NP}$-complete, show that Max-TSP is also $\mathcal{NP}$-complete.

Find a polynomial time 1/2-approximation algorithm for a Max-TSP optimization problem, that is, an algorithm that produces a tour with length no less than 1/2 the optimum. Hint: obtain an upper bound for the Max-TSP problem from the solution to an assignment problem, and then modify this solution so that a single tour is obtained.

6 Define the notion of a Nash equilibrium in a n-person, nonzero-sum game.

A 'symmetric game' is one in which the same strategies are available to all players and the payoff that a player obtains when playing a particular pure strategy depends only that strategy and the strategies that other players employ, not on the identities of who plays them. A symmetric equilibrium is one in which all players use the same strategy, possibly mixed. Let $e(i, p)$ be the expected payoff to a player who plays pure strategy i against opponents who independently each use the same mixed strategy, which randomizes over k pure strategies with probabilities $p = (p_1, \dots, p_k)$. Let $e(p) = \sum_{i=1}^{k} p_i e(i, p)$. Prove that at least one symmetric equilibrium is guaranteed to exist. You may assume the Brouwer fixed point theorem.

In a 'least unique bid auction' bidders are required to make their bids from a set of values, say $\{1, 2, \dots, k\}$ and the winner is the one who makes the least unique bid. The winner pays his bid and obtains the object, which is worth V. If there is no unique bid then there is no winner. Consider such an auction, with 3 bidders, $k = 2$ and $V = 3$. Find all the symmetric equilibria.

What is the number of nonsymmetric equilibria?

END OF PAPER