MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 1.30 to 3.30

PAPER 36

STOCHASTIC LOEWNER EVOLUTIONS

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let K be a compact $\mathbb{H}$-hull and set $H = \mathbb{H} \setminus K$. Show that there exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.

(b) Suppose that K_0 is a compact $\mathbb{H}$-hull and $g_{K_0}(z) = z + z^{-1}$ for all $z \in \mathbb{H} \setminus K_0$. Identify K_0 and find $\operatorname{hcap}(K_0)$.

(c) Suppose that $|z| \leq 1$ for all $z \in K$. Show that, for all $x \in \mathbb{R}$ with $|x| > 1$,

$$1 - x^{-2} \leq g_K'(x) \leq 1.$$

[In (c), you may use without proof any result from the course.]

2 (a) Let $(\gamma_t)_{t \geq 0}$ be an SLE(κ), for some $\kappa \in [0, \infty)$. Explain the relation to $(\gamma_t)_{t \geq 0}$ of the associated Loewner flow $(g_t)_{t \geq 0}$ and transform $(\xi_t)_{t \geq 0}$.

(b) Fix $s \geq 0$ and define for $t \geq 0$

$$\bar{\gamma}_t = g_s(\gamma_{s+t}), \quad \tilde{\gamma}_t = \bar{\gamma}_t - \xi_s.$$

What is the Loewner transform of $(\bar{\gamma}_t)_{t \geq 0}$? What is the distribution of $(\tilde{\gamma}_t)_{t \geq 0}$? Justify your answers.

(c) Suppose now that $\kappa \in (0, 4]$. Show that, almost surely, $(\gamma_t)_{t \geq 0}$ is a simple curve. [You may assume without proof that, almost surely, $\operatorname{Im}(\gamma_t) > 0$ for all $t > 0$.]

3 (a) Let γ be an SLE(8/3). Let U be a simply connected domain in the upper half-plane $\mathbb{H}$, which is a neighbourhood of both 0 and ∞. Denote by Φ the unique conformal isomorphism $U \to \mathbb{H}$ such that $\Phi(z) - z \to 0$ as $z \to \infty$. Set $K_t = \{\gamma_s : 0 < s \leq t\}$ and $T = \inf\{t \geq 0 : \gamma_t \notin U\}$. Define, for $t < T$,

$$K_t^* = \{\Phi(\gamma_s) : 0 < s \leq t\}, \quad \Phi_t = g_{K_t^*} \circ \Phi \circ g_{K_t}^{-1},$$

where, for K a compact $\mathbb{H}$-hull, $g_K : (\mathbb{H} \setminus K) \to \mathbb{H}$ is the unique conformal isomorphism such that $g_K(z) - z \to 0$ as $z \to \infty$. Set

$$\Sigma_t = \Phi_t'(\xi_t),$$

where ξ is the Loewner transform of γ. Show that a suitably chosen function of the process Σ is a local martingale.

(b) Hence, show that

$$\mathbb{P}(\gamma_t \in U \text{ for all } t \geq 0) = \Phi'(0)^{5/8}.$$

[You may assume without proof any standard identities of the classical Loewner theory, or for the Brownian excursion. You may also assume that $\Sigma_t \to 1_{\{T=\infty\}}$ as $t \uparrow T$, almost surely.]

4 (a) Let μ be a scale-invariant probability measure on chords in the upper half-plane from 0 to ∞. What does it mean to say that μ has the locality property?

(b) Let γ be an SLE(6) and let $\Phi : N \to N^*$ be a conformal isomorphism of one neighbourhood of 0 in $\mathbb{H}$ to another. Assume that $\Phi(0) = 0$ and that $\Phi(\bar{N} \cap \mathbb{R}) = \overline{N^*} \cap \mathbb{R}$. Set $T = \inf\{t \geq 0 : \gamma_t \notin N\}$ and define, for $t < T$,

$$\gamma_t^* = \Phi(\gamma_t).$$

Write $(\xi_t^*)_{t<T}$ for the Loewner transform of $(\gamma_t^*)_{t<T}$. Show that $(\xi_t^*)_{t<T}$ is a local martingale.

(c) Deduce that the law of $[\gamma]$ has the locality property.

END OF PAPER