MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 1.30 to 3.30

PAPER 33

MODULAR AND AUTOMORPHIC FORMS

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Λ be a lattice in $\mathbb{C}$, with basis $\{\omega_1, \omega_2\}$. Define the Weierstraß $\wp$-function associated to Λ, and show that is an elliptic function with respect to Λ. [*You may assume without proof the convergence of the series* $\sum' |\omega|^{-\sigma}$ *for* $\sigma > 2$.]

Compute the Laurent series of $\wp(z)$ at the origin in terms of the constants

$$G_k(\Lambda) = \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^k}.$$

Show that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4 \prod_{i=1}^{3} (\wp(z) - e_i)$$

where $e_i = \wp(\omega_i/2)$ and $\omega_3 = -\omega_1 - \omega_2$.

Prove that

$$\frac{\wp'(z - \frac{1}{2}\omega_i)}{\wp'(z)} = -\left(\frac{\wp(\frac{1}{4}\omega_i) - e_i}{\wp(z) - e_i} \right)^2 .$$

2 What is a modular form of weight k? Show that for $k > 2$ the holomorphic Eisenstein series

$$G_k(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^k}$$

is a modular form of weight k with q-expansion

$$G_k(\tau) = 2\zeta(k) \left(1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \right)$$

where the Bernoulli numbers B_k are defined by the identity

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Stating clearly any results you use, show that the Fourier coefficients $\tau(n)$ of the cusp form Δ satisfy the congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

[*The equality* $B_{12} = -691/2730$ *may be useful.*]

3 Write an essay on the theory of Hecke operators for modular forms on $SL_2(\mathbb{Z})$.

4 The real analytic Eisenstein series is defined for $\mathrm{Re}(s) > 1$, $\tau = x + iy \in \mathcal{H}$ as

$$E(\tau, s) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|m\tau + n|^{2s}} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\, \gamma(\tau))^s$$

Let f, $g \in S_k$ be cusp forms with q-expansions

$$f(\tau) = \sum_{n \geq 1} a_n q^n, \qquad g(\tau) = \sum_{n \geq 1} b_n q^n.$$

Show that for $\mathrm{Re}(s)$ sufficiently large, the Dirichlet series

$$D(f, g, s) = \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^s}$$

can be computed in terms of the Rankin–Selberg integral

$$I(f, g, s) = \int_{\Gamma \backslash \mathcal{H}} f(\tau) \overline{g(\tau)} E(\tau, s) y^{k-2}\, dx\, dy.$$

Assuming any analytic results concerning $E(\tau, s)$ you need, show that $D(f, g, s)$ can be analytically continued to the half-plane $\{s \mid \mathrm{Re}(s) > k\}$. Hence show that $D(f, f, s)$ is absolutely convergent for $\mathrm{Re}(s) > k$, and deduce that the L-series $L(f, s)$ converges absolutely for $\mathrm{Re}(s) > (k+1)/2$.

[*The inequality* $|a_n| \leq \max(n^\alpha, |a_n|^2 / n^\alpha)$ *may be useful.*]

END OF PAPER