UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 3.30

PAPER 32

ALGEBRAIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

ζ_n denotes a primitive n^{th} root of unity. O_K denotes the ring of integers of K.

1

(i) State the Kummer-Dedekind theorem.

(ii) Let $L = \mathbb{Q}(\alpha)$, where α is a root of the monic irreducible polynomial $f(X) \in \mathbb{Z}[X]$. Suppose p is a prime number such that $f(X)$ mod p has no repeated roots in the algebraic closure of $\mathbb{F}_p$. Prove that the index $[O_L : \mathbb{Z}[\alpha]]$ is coprime to p.

(iii) Determine which primes ramify in $\mathbb{Q}(\sqrt[11]{44})/\mathbb{Q}$. Justify your answer.

2

(i) Let F/K be a Galois extension of number fields and $\mathbf{p}$ a prime of K. Prove that the Galois group $\mathrm{Gal}(F/K)$ acts transitively on the set of primes of F above $\mathbf{p}$. Explain briefly how this may be used to determine the number of primes above $\mathbf{p}$ in an intermediate extension $K \subset L \subset F$, in terms of the decomposition group of a prime above $\mathbf{p}$ in F/K.

(ii) Let $F = \mathbb{Q}(\zeta_5, \sqrt[5]{\lambda})$ for some prime number λ. Let $\mathbf{q}$ be a prime of F above a prime p of $\mathbb{Q}$, whose decomposition group in $\mathrm{Gal}(F/\mathbb{Q})$ is cyclic of order 2. Show that there are three primes above p in $\mathbb{Q}(\sqrt[5]{\lambda})$, and two primes above p in $\mathbb{Q}(\zeta_5)$.

3 Define the Dirichlet L-function $L_N(\psi, s)$ for a Dirichlet character ψ modulo N, and state its expression as an Euler product. Prove that if ψ is non-trivial, then $L_N(\psi, s)$ is analytic on $\mathrm{Re}(s) > 0$ and that $L_N(\psi, 1) \neq 0$.

Prove Dirichlet's theorem on primes in arithmetic progressions. You may assume that for a Dirichlet character ψ,

$$\sum_{p \text{ prime},\, n \geq 1} \frac{\psi(p)^n}{n}\, p^{-ns}$$

converges absolutely on $\mathrm{Re}(s) > 1$ to an analytic branch of the logarithm of $L_N(\psi, s)$.

(*Standard results on convergence of Dirichlet series may be used without proof. You may also assume that the Riemann ζ-function has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s = 1$.*)

4

Let $F = \mathbb{Q}(\zeta_3, \sqrt[3]{3})$, and let ρ be the two-dimensional irreducible representation of $\mathrm{Gal}(F/\mathbb{Q}) \simeq S_3$. Compute the first ten coefficients $a_1, \ldots, a_{10}$ of its Artin L-series $L(\rho, s) = \sum_n a_n n^{-s}$.

END OF PAPER