MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 1.30 to 4.30

PAPER 3

LIE ALGEBRAS AND REPRESENTATION THEORY

Attempt no more than ***THREE*** *questions, which* **must** *include the first question.*

There are ***FIVE*** *questions in total.*

The 2nd - 5th questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangle and squared paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let V be the defining representation for $\mathfrak{sl}_2$. Let

$$U = \wedge^2 Sym^4 V.$$

(i) Write out a basis for U. Draw the weight diagram for U, and give explicit bases for each weight space of U.

(ii) Which irreducible modules occur as submodules of U? Find a highest weight vector for a submodule $W$of dimension 3.

(iii) Give a non-zero element in W_0.

(b) Let W be the defining representation for $\mathfrak{sl}_3$. Let

$$T = Sym^2 W \otimes Sym^2 W\,.$$

(i) Describe a basis for T (or write it out in full). Draw the weight diagram for T on a sheet of triangle paper.

(ii) For each weight λ of T which lies in the closure of the fundamental Weyl chamber give explicitly a basis for T_λ.

(iii) Find a highest weight vector in $T_{2L_1-L_3}$. If Z is the submodule generated by this vector, find a basis for Z_{L_1}.

(c)

(i) Suppose that a representation W of $\mathfrak{sl}_3$ is isomorphic to the direct sum of two irreducible submodules, $W = U \oplus V$. Suppose also that $\dim W_{L_1-2L_3} = 3$. If $U_{L_1-2L_3}$ and $V_{L_1-2L_3}$ are both non-zero list all candidate pairs U, V of irreducible $\mathfrak{sl}_3$ modules satisfying $W = U \oplus V$, and $\dim W_{L_1-2L_3} = 3$.

(ii) If additionally $\dim W_{-L_3} = 5$ which candidates for U, V are still possibilities? (Hint: plot $L_1 - 2L_3$ on the weight lattice.)

2

Let $\mathfrak{g}$ be a semi-simple Lie algebra, and let R be the set of roots. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

be a Cartan decomposition of $\mathfrak{g}$.

(i) Define the Weyl group $\mathfrak{W}$ of $\mathfrak{g}$. Results about semi-simple Lie algebras may be used if stated clearly.

(ii) Explain what is meant by the fundamental Weyl chamber $\mathcal{W}$. Again, constructions and results may be used without proof, but should be stated clearly.

(iii) What is meant by the highest weight of a representation? Show the existence of a highest weight for any finite dimensional representation.

(iv) Show that if λ is a highest weight of a representation, then λ is in the closure $\overline{\mathcal{W}}$ of $\mathcal{W}$.

(v) Find the element of $\mathfrak{W}$ that sends $-2L_1 + L_2$ to an element μ in the fundamental Weyl chamber. What is μ? You may use the weight diagram to get the answer, but the answer should be confirmed by computation.

3

Let $\mathfrak{so}_5$, the special orthogonal Lie algebra, be given explicitly as the Lie subalgebra of $\mathfrak{gl}_5$ which preserves the matrix M.

$$\mathrm{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Give the general form of an element Z in $\mathfrak{so}_5$.

(ii) If $\mathfrak{h}$ is the set of diagonal elements in $\mathfrak{so}_5$, and $L_i : \mathfrak{h} \to \mathbb{C}$ is the usual linear map sending a diagonal matrix to the iith entry, give the Cartan decomposition of $\mathfrak{so}_5$ explicitly, identifying a basis element X_α for each root α.

(iii) On the squared paper provided draw the weight diagrams for (a) the defining representation and (b)the adjoint representation of $\mathfrak{so}_5$.

(iv) Define the Killing form $(*, *)$ for a Lie algebra. Show that for H, K in $\mathfrak{h}$

$$(H, K) \ = \sum_\alpha \alpha(H)\, \alpha(K)\,.$$

(v) Compute (H,H) where H is the matrix

$$\mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4

(i) Explain what is meant by an admissable Coxeter graph.

(ii) Show that if Γ is an admissable graph, then Γ contains no cycles.

(iii) Let $\mathfrak{g}$ be a Lie algebra with a given Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha root} \mathfrak{g}_\alpha$, and an ordering on the weight space which determines an ordering on the roots. Explain how the above information determines a Dynkin diagram. Facts about roots α and root spaces $\mathfrak{g}_\alpha$ may be stated without proof.

(iv) Show that if α_1, α_2 are simple roots, then $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}] = 0$ if and only if the corresponding nodes are not connected by an edge.

(v) If $\alpha_1, \alpha_2, \alpha_3$ are three simple roots, conclude that $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_j}] = 0$ for some $\{i, j\} \subset \{1, 2, 3\}$.

5

(i) Let $X \in M_n(\mathbb{C})$, and let $X = X_s + X_n$ be the decomposition of X into its semi-simple and nilpotent parts. What properties characterize this decomposition? If X is the matrix

$$\mathrm{X} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

find X_s and X_n.

(ii) Show that if $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a semi-simple Lie alebra and X is in $\mathfrak{g}$, then the semi-simple and nilpotent parts of X are also in $\mathfrak{g}$. Results about the Jordan decomposition of a matrix and representations of semi-simple Lie algebras may be assumed but should be stated clearly.

(iii) Explain in no more than five sentences what the absolute Jordan decomposition is, and the implications of this decomposition for the theory of representations of semi-simple Lie algebras.

(iv) Find an example of a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_2$ and an element X in $\mathfrak{g}$ such that X_s, X_n are not in $\mathfrak{g}$.

END OF PAPER