MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 1.30 to 3.30

PAPER 29

LOCAL FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

$\mathbb{Q}_p$ denotes the field of p-adic numbers, $\mathbb{Z}_p$ the ring of p-adic integers and $k((t))$ the ring of formal Laurent series over k.

1 Let K be a non-Archimedean local field with residue field k.

(a) State and prove a version of Hensel's lemma.

(b) Prove the existence of the Teichmuller lift $k^* \hookrightarrow K^*$.

(c) If $\mathrm{char}(K) > 0$, deduce that $K \cong k((t))$.

State all the results that you use.

2 (a) Define the notion of a non-Archimedean absolute value and when two absolute values are equivalent. Classify, with proof, all non-trivial non-Archimedean absolute values on $\mathbb{Q}$ up to equivalence.

(b) Give a basis of open sets for the topology on $\mathbb{Z}_p$. Show that a subgroup of $(\mathbb{Z}_p, +)$ is open if and only if it is of finite index.

3 Determine the smallest $k \geq 1$ such that for all $u \in \mathbb{Z}_2^*$,

$$\sqrt{u} \in \mathbb{Z}_2 \quad \Longleftrightarrow \quad u \equiv 1 \bmod 2^k,$$

proving all the assertions that you make. Deduce the structure of $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, and write down all seven quadratic extensions of $\mathbb{Q}_2$. Which ones are unramified? For one of the remaining ones, compute the ramification groups G_i for all $i \geq 1$.

4 (a) Suppose K is a non-Archimedean local field, and let L/K be a finite Galois extension with Galois group G. What is meant by the inertia and the wild inertia subgroups of G? What is meant by tame inertia? Prove that the tame inertia group is cyclic. (You may use without proof that $O_L = O_K[\pi_L]$ if L/K is totally ramified.)

(b) Prove that a Galois extension of $\mathbb{C}((t))$ of degree n is isomorphic to $\mathbb{C}((\sqrt[n]{t}))$, and determine $\mathrm{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t)))$.

END OF PAPER