UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 1.30 to 4.30

PAPER 26

CATEGORY THEORY

You should attempt **one** *question from Section 1, and* **two** *from Section 2.*

There are **six** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION 1

1 Explain what is meant by the terms *monad*, *Eilenberg–Moore category* and *monadic adjunction*. State and prove the Precise Monadicity Theorem, and use it to show that the adjunction formed by the forgetful functor from the category of compact Hausdorff spaces to **Set**, and its left adjoint, is monadic.

[Standard results from general topology, and the existence of the left adjoint, may be assumed.]

2 It has been said that category theory is the one part of mathematics where definitions matter more than theorems. Write a short essay arguing the case *either* for *or* against this assertion, illustrating your argument with examples drawn from the course.

SECTION 2

3 Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Explain what is meant by the arrow category $(A \downarrow F)$, where A is a fixed set.

Show that F may be expressed as the colimit of a diagram of shape $(1 \downarrow F)^{\mathrm{op}}$ in $[\mathcal{C}, \mathbf{Set}]$ (where 1 denotes a one-element set) whose vertices are representable functors. Deduce that if $\mathcal{C}$ has finite limits, the following conditions are equivalent:

(i) F preserves finite limits.

(ii) For any set A, $(A \downarrow F)$ has finite limits.

(iii) $(1 \downarrow F)^{\mathrm{op}}$ is filtered.

[You may assume the result that filtered colimits commute with finite limits in **Set**.]

4 Define a *balanced* category. If $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor and $\mathcal{C}$ is balanced, prove that F reflects isomorphisms.

Let $((F : \mathcal{C} \to \mathcal{D}) \dashv (G : \mathcal{D} \to \mathcal{C}))$ be an adjunction with unit η and counit ϵ. Show that F is faithful if and only if η is a (pointwise) monomorphism. Now suppose that $\mathcal{C}$ is balanced, and that every morphism of $\mathcal{D}$ can be factored as a regular epimorphism followed by a monomorphism. Show that the following are equivalent:

(i) Both η and ϵ are monomorphisms.

(ii) F is full and faithful, and its image is closed (up to isomorphism) under regular quotients in $\mathcal{D}$. (That is, if $FA \to B$ is regular epic, then B is isomorphic to some FA'.)

Give an example of an adjunction whose unit and counit are both monic, but whose left adjoint is not full.

5 Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a full subcategory of the functor category $[\mathcal{C}, \mathcal{C}]$ which is closed under composition and contains the identity functor. Suppose $\mathcal{D}$ has a terminal object T: show that T carries a unique monad structure $\mathbf{T}$, and that if $\mathbf{S}$ is any monad on $\mathcal{C}$ whose functor part lies in $\mathcal{D}$ then there is a forgetful functor $\mathcal{C}^{\mathbf{T}} \to \mathcal{C}^{\mathbf{S}}$.

Now let $\mathcal{C} = \mathbf{Set}$, and let $\mathcal{D}$ be the category of all functors $\mathbf{Set} \to \mathbf{Set}$ which preserve finite coproducts. Show that $\mathcal{D}$ has a terminal object T, and that TA may be identified with the set of all ultrafilters on A.

[Recall that an *ultrafilter* on a set A is a family $\mathcal{F}$ of subsets of A satisfying $(B \in \mathcal{F}, B \subseteq C \Rightarrow C \in \mathcal{F})$, $(B \in \mathcal{F}, C \in \mathcal{F} \Rightarrow (B \cap C) \in \mathcal{F})$ and (for all $B \subseteq A$, exactly one of B and $A \setminus B$ is in $\mathcal{F}$).]

6 Define the notions of *abelian category* and of *exact sequence* in an abelian category. Prove that evey morphism in an abelian category may be factored as an epimorphism followed by a monomorphism. Show also that, in an exact sequence

$$0 \to A \to B \to C \to 0 \ ,$$

the morphism $A \to B$ is split monic if and only if $B \to C$ is split epic.

Let $\mathcal{A}$ be an abelian category with enough projectives (i.e., such that every object admits an epimorphism from a projective object). Sketch the construction of the left derived functors $L^n F$ of a right exact functor $F\mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is any abelian category, and write down the long exact sequence in $\mathcal{B}$ induced by an exact sequence $(0 \to A \to B \to C \to 0)$ in $\mathcal{A}$. [Detailed proofs are not required: in particular, you need not prove that the $L^n F$ are well-defined or functorial.]

Show that an object A of $\mathcal{A}$ is projective if and only if $L^1 FA = 0$ for all right exact $F \colon \mathcal{A} \to \mathcal{B}$. [Hint: given A, consider the functor $\mathcal{A}(-, K) \colon \mathcal{A} \to \mathbf{AbGp}^{\mathrm{op}}$, where $(0 \to K \to P \to A \to 0)$ is an exact sequence with P projective.] Deduce that the following conditions on $\mathcal{A}$ are equivalent:

(i) Every subobject of a projective object is projective.

(ii) For any right exact $F \colon \mathcal{A} \to \mathcal{B}$, $L^2 F$ is identically 0.

(iii) For any right exact $F \colon \mathcal{A} \to \mathcal{B}$, $L^1 F$ is left exact.

END OF PAPER