UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 1.30 to 4.30

PAPER 25

ALGEBRAIC CYCLES

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let E be a complex elliptic curve. Show that the curves $E \times \text{point}$, $\text{point} \times E$, and the diagonal $\Delta \cong E$ are linearly independent in $H^2(E \times E, \mathbf{Q})$. Deduce that the Künneth formula fails for Chow groups, in the sense that the natural homomorphism

$$CH^*(E) \otimes_{\mathbb{Z}} CH^*(E) \to CH^*(E \times E)$$

given by $x \otimes y \mapsto \pi_1^*(x)\pi_2^*(y)$ is not surjective. Here π_1 and π_2 denote the two projections $E \times E \to E$.

2 Let $f : X \to Y$ be a surjective morphism of complex projective varieties of dimension 2, with X smooth over $\mathbb{C}$ and Y normal. Suppose that there is exactly one irreducible curve C in X which f maps to a point. Show that the self-intersection number C^2 is negative. [Hint: Embed Y in some projective space, and consider hyperplane sections of Y.]

3 **(a)** Let X be a smooth complex projective surface. Let C be an irreducible curve on X with negative self-intersection. Show that any effective divisor linearly equivalent to C is equal to C.

(b) Show that a smooth complex projective surface contains at most countably many irreducible curves with negative self-intersection. Can a smooth complex projective surface contain uncountably many irreducible curves with zero self-intersection?

(c) Using (b) or otherwise, show that a smooth surface of degree at least 3 in $\mathbf{P}^3$ contains at most finitely many lines.

4 Let X be a complex abelian surface (a compact complex torus of complex dimension 2 which is also a projective variety). Show that there is a surjective homomorphism of algebraic groups from the Jacobian of some smooth curve onto X. [Hint: Choose a projective embedding of X, and consider a hyperplane section of X.]

5 **(a)** Let C, D be effective divisors of degree 3 in $\mathbf{P}^2$ over $\mathbb{C}$. Suppose that $C \cap D$, as a set, is just a single point p. What can you say about the intersection multiplicity of C and D at p?

(b) Give an example of a triple line $C = 3L$ (for some line L in $\mathbf{P}^2$) and an irreducible cubic curve D which is smooth at a point p such that $C \cap D$, as a set, is the single point p.

(c) Classify pairs C, D as in **(b)** such that, in addition, D is a cuspidal cubic. (That means that D is projectively isomorphic to the curve $y^2z = x^3$.) "Classify" here means up to projective isomorphism, that is, up to the action of $PGL(3, \mathbf{C})$ on the set of pairs $C, D \subset \mathbf{P}^2$.

6 **(a)** Let X be a smooth complex projective surface of degree d in $\mathbf{P}^3$. Suppose that X contains a smooth rational curve C which has degree a (as a curve in $\mathbf{P}^3$). Compute the self-intersection number of C in X.

(b) Show that a very general complex surface X of degree d at least 4 in $\mathbf{P}^3$ contains no smooth rational curve. (You may use theorems from the course.)

END OF PAPER