UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 1.30 to 4.30

PAPER 22

COMPLEX MANIFOLDS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the differential (p, q)-forms on a complex manifold and the differential operators ∂ and $\bar{\partial}$. Explain what is meant by a real (p, p)-form. For a real (p, p)-form η, show that $d\eta = 0$ if and only if $\bar{\partial}\eta = 0$. Show also that if $f = u + iv$ is a holomorphic function on a complex manifold then its real part u satisfies $\bar{\partial}\partial u = 0$.

Suppose that φ is a $(0, q)$-form on a polydisc U in $\mathbb{C}^n$, $q > 0$, and $\bar{\partial}\varphi = 0$. Show that there is a $(0, q-1)$-form ψ defined on an open subset $U_0 \subset U$ such that $\bar{\partial}\psi = \varphi|_{U_0}$.

By applying the latter result on suitable neighbourhoods, but without appealing to Hodge theory, deduce that every $(0, 1)$-form on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is $\bar{\partial}$-exact.

[*You may assume that if $D \subseteq \mathbb{C}$ is open and a complex function g is smooth on D then, for a closed disc $\overline{D}_0 \subset D$, the function $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{D_0} \dfrac{g(w)}{w - z}\, dw\, d\bar{w}$ is smooth and satisfies $\dfrac{\partial f}{\partial \bar{z}} = g$ on the interior D_0 of $\overline{D}_0$.*

At some point, you might like to consider the Laurent expansion of an appropriate holomorphic function on an annulus in $\mathbb{C}$.]

2 Define the terms *holomorphic line bundle* L over a complex manifold X, *holomorphic section* of L over an open subset $U \subseteq X$ and the *dual* bundle L^* of L, showing that L^* is a holomorphic bundle.

Now suppose that X is compact and connected. Show that L is holomorphically trivial if and only if both L and L^* have non-identically-zero holomorphic sections over X.

Define the tautological bundle $\mathcal{O}(-1)$ and the hyperplane bundle $\mathcal{O}(1)$ over $\mathbb{C}P^n$ and show that these bundles have holomorphic transition functions. Let H be a hyperplane in $\mathbb{C}P^n$. Give an example of a never-zero holomorphic section of $\mathcal{O}(1)$ over $\mathbb{C}P^n \setminus H$ which extends holomorphically over H.

[*Standard results about complex vector bundles over smooth manifolds may be assumed, provided these are accurately stated. You may also assume standard properties of holomorphic functions on open neighbourhoods in $\mathbb{C}^n$.*]

3 Define the terms *divisor*, *order of a meromorphic function* at an irreducible hypersurface and *principal divisor* on a complex manifold. You should state the auxiliary properties of local rings that you require. Explain what is meant by a local defining function for a divisor D and by the holomorphic line bundle $[D]$ associated to D. Show that $[D]$ is holomorphically trivial if and only if D is a principal divisor.

State the adjunction formula for the canonical bundle of a non-singular hypersurface in a complex manifold.

Let Q_1 and Q_2 be complex homogeneous polynomials on $\mathbb{C}^5$ of degrees, respectively, d_1 and d_2. Suppose that the zero locus W of Q_1 defines a non-singular connected hypersurface in $\mathbb{C}P^4$ and the common zero locus of Q_1 and Q_2 defines a non-singular complex surface S in $\mathbb{C}P^4$. Determine all the values of d_1,d_2, such that S has a trivial canonical bundle.

[*You may assume that the canonical bundle of* $\mathbb{C}P^n$ *is isomorphic to* $[-(n+1)H]$, *where* H *is a hyperplane, and that* $S \cap H_0$ *is non-empty for some hyperplane* H_0 *in* $\mathbb{C}P^4$. *The relation* $[D_1 + D_2] = [D_1] \otimes [D_2]$ *for divisors* D_1, D_2 *may be used without proof.*]

4 Define the fundamental $(1,1)$-form ω of a Hermitian metric on a complex manifold. Explain briefly how the volume form of the induced Riemannian metric is expressed in terms of ω.

Define the Hodge $*$-operator for complex differential forms on a Hermitian manifold. Show that on a Hermitian manifold of (complex) dimension n every $(n,0)$-form η satisfies $*\eta = c\eta$, with $c = (-1)^{n(n+1)/2}\, i^n$.

Show that the operator $\bar{\partial}^* = -*\partial*$ is the formal L^2 adjoint of $\bar{\partial}$. Define $\bar{\partial}$-harmonic forms and state the Hodge theorem for the space of (p,q)-forms. Show that the space of $\bar{\partial}$-harmonic (p,q)-forms on a compact Hermitian manifold X is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$ and also to the space of $\bar{\partial}$-harmonic $(n-p, n-q)$-forms $(n = \dim_{\mathbb{C}} X)$.

[*You may assume that* $**\,\alpha = (-1)^{p+q}\alpha$ *for each* (p,q)*-form* α.]

5 Let X be a compact Kähler manifold. Define the Laplacians Δ, Δ_{∂}, $\Delta_{\bar{\partial}}$, Δ^c corresponding to, respectively $d, \partial, \bar{\partial}, d^c$. Prove the identities $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial} = \Delta^c$.

Show that a complex differential form α on X satisfies $\Delta\alpha = 0$ if and only if $\Delta(J(\alpha)) = 0$, where J denotes the almost complex structure on X.

State the $\partial\bar{\partial}$-lemma for X. Show that if ω and $\tilde{\omega}$ are two Kähler forms in the same de Rham cohomology class then $\tilde{\omega} = \omega + i\partial\bar{\partial}f$ for some smooth real-valued function f on X and f is uniquely determined up to additive constant.

[*You may assume the identity* $[\Lambda, \partial] = i\bar{\partial}^*$ *on a Kähler manifold, provided that you give a definition of* Λ.]

END OF PAPER