UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2008 1.30 to 4.30

PAPER 20

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with the standard ('round') metric. Denoting the Laplacian on $\mathcal{C}^\infty(\mathbb{R}^{d+1})$ by $\tilde{\Delta}$, the Laplacian on $\mathcal{C}^\infty(S^d)$ by Δ and the restriction to S^d of $f \epsilon\, \mathcal{C}^\infty(\mathbb{R}^{d+1})$ by $\bar{f}$, state the relation between $\overline{\tilde{\Delta} f}$ and $\Delta\bar{f}$.

Obtain the spectrum of Δ, specifying the multiplicity of each eigenvalue.

2 Given the two-dimensional subspace T of $(\mathbb{F}_3)^4$ whose non-zero vectors are

$$\pm(0,-1,-1,-1), \pm(1,0,1,-1), \pm(1,-1,0,1), \pm(1,1,-1,0),$$

describe how to construct pairs of lattices in $\mathbb{R}^4$ with the same length spectrum.

For one such pair of lattices prove that they are not isometric.

Deduce the existence of a pair of isospectral non-isometric flat 4-tori, quoting without proof any lemmas that you require.

3 Define a nowhere homogenous or 'bumpy' metric on a differential manifold M^m and state Sunada's Lemma.

State the dimension of the k-jet space for maps from M^m to another differential manifold N^n, and identify its fibre as a bundle over $M \times N$.

Given a countable basis $\{U_i \mid i \epsilon N\}$ of open sets of M^m such that each $\bar{U}_i$ is diffeomorphic with a closed m-ball, let $\mathcal{S}_{ik}$ be the set of metrics on M such that there is an isometry $\phi : \bar{U}_i \to \bar{U}_k$. Prove that the complement $\mathcal{C}\mathcal{S}_{ik}$ of $\mathcal{S}_{ik}$ is dense in the space of all metrics on M with the $\mathcal{C}^\infty$-topology.

4 If T is a finite group define what it means for subgroups U and V to be Gassman equivalent.

Assuming that T may be generated by two elements A and B describe how to construct a surface M on which T acts as a group of homeomorphisms. What is $\chi(M)$ and what constraints are required to ensure that $M_1 = U\backslash M$ and $M_2 = V\backslash M$ are normally covered by M.

State Sunada's trace formula and explain how this enables you to choose Riemmanian metrics on M_1 and M_2 such that they are isospectral.

State a necessary condition on U and V for M_1 and M_2 to not be isometric and state how to ensure that this is also sufficient.

Specify the parameters that will produce pairs of isospectral non-isometric surfaces of each of the genera 2 and 3.

5 Define the Teichmüller space for closed Riemann surfaces of genus g, stating sufficient facts about Riemann surfaces to make this comprehensible.

State Wolpert's Theorem, Huber's Theorem and the Theorem of Buser that simplifies Wolpert's proof.

Prove Buser's Theorem stating clearly, but without proof, any subsidiary results that you require.

END OF PAPER