UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2008 1.30 to 4.30

PAPER 18

ALGEBRAIC TOPOLOGY

*Attempt at most **FOUR** questions.*

*There are **SEVEN** questions in total.*

All questions are of equal weight. Standard theorems from the course may be used without proof if carefully stated. Knowledge of the (co)homology of spheres can be assumed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the Excision theorem and deduce the Mayer-Vietoris theorem for singular homology. Use these to compute the homology groups (with $\mathbb{Z}$-coefficients) of $\mathbb{RP}^1 \times \mathbb{RP}^2$.

Is this space homotopy equivalent to any oriented manifold? Briefly justify your answer.

2 What does it mean to say a connected topological space X is simply connected? State Whitehead's theorem concerning homotopy equivalences of simply connected finite cell complexes. You may henceforth assume that any map of finite cell complexes is homotopic to a cellular map (one which takes k-skeleta to k-skeleta).

Compute the cellular cohomology of the space obtained from a circle by attaching two 2-cells by maps of degrees 2 and 3. Show that this space is homotopy equivalent to the 2-sphere. Is it homeomorphic to the 2-sphere?

Give two compact spaces which have additively isomorphic cohomology but which are not homotopy equivalent, briefly justifying your example.

3 Define the cup-product on cohomology and state the Künneth theorem. Let Σ_g denote a closed oriented surface of genus g. Compute $H^*(\Sigma_g; \mathbb{Z})$ as a ring. Interpret Poincaré duality in terms of intersections of submanifolds, and illustrate this for $H^*(\Sigma_2; \mathbb{Z})$ and $H^*(\Sigma_2 \times \Sigma_2; \mathbb{Z})$.

If $f : \Sigma_{g_1} \times \Sigma_{g_2} \to \Sigma_{g_1} \times \Sigma_{g_2}$ is homotopic to the identity and has no fixed points, what is $g_1 g_2$? Briefly justify your answer.

4 State the Poincaré duality theorem. Let M be a closed connected 3-manifold with first Betti number b, so $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^b \oplus \langle \text{Tors} \rangle$ where $\langle \text{Tors} \rangle$ denotes the torsion subgroup. Show that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^b$ if M is orientable and is $\mathbb{Z}^{b-1} \oplus \mathbb{Z}/2$ if M is not orientable. Deduce that if M is not orientable then it has positive first Betti number.

Let the graded group $K = \oplus_{j=0}^{6} K_j$ be obtained from H by setting $K_j = H_{j \bmod 3}$, where H is the graded cohomology group of a closed connected orientable 3-manifold. Is there necessarily a closed 6-manifold with graded cohomology group equal to K? Can K ever be the graded cohomology group of a closed 6-manifold? Briefly justify your answers.

5 Define the degree of a map $f : M \to N$ between oriented closed manifolds M and N of the same dimension. Suppose for some $n \in N$, the set $f^{-1}(n)$ is finite. Associate to each point in this set a local degree and explain how these relate to the degree of f.

(i) Show that the local degree can take any integer value.

(ii) Construct a surjective degree zero map $S^2 \to T^2$ or $T^2 \to S^2$.

(iii) Show any closed oriented manifold has a degree one map to the sphere.

(iv) For each k show there is some natural number $g(k)$ and a degree k map $\Sigma_{g(k)} \to \Sigma_2$.

6 What is a vector bundle? Define the Grassmannian $Gr_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ and the tautological bundle $E_{taut} \to Gr_k(\mathbb{R}^n)$. Let Q_{taut} denote the quotient of the trivial rank n bundle over $Gr_k(\mathbb{R}^n)$ by the tautological bundle E_{taut}: explain why Q_{taut} is a vector bundle.

Let M be a compact manifold and $E \to M$ a vector bundle of rank d. Assume that for every $m \in M$ and $v \in E_m$ there is a section s of E such that $s(m) = v$. Show that there is a continuous map $\phi : M \to Gr_{N-d}(\mathbb{R}^N)$, for some N, such that $E = \phi^* Q_{taut}$. Does the quotient bundle Q_{taut} always have a nowhere-zero section?

7 State the Thom Isomorphism theorem and deduce the Gysin exact sequence for the cohomology of the sphere bundle of a vector bundle. Using this, compute $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ as a ring.

Prove that every map $f : \mathbb{RP}^4 \to \mathbb{RP}^4$ has a fixed point. Show there is a map $\mathbb{RP}^3 \to \mathbb{RP}^3$ without fixed points.

END OF PAPER