MATHEMATICAL TRIPOS **Part III**

Friday 6 June 2008 9.00 to 11.00

PAPER 15

COMBINATORIAL PROBABILITY

Attempt the FIRST QUESTION and ANY TWO other questions.

The FIRST QUESTION carries 40% of the marks, and the other questions carry 30% each.

Any results you quote should always be stated precisely.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{A}$ be a subset of the n-dimensional discrete cube $Q^n = \{0,1\}^n$. Write $t = |\mathcal{A}|/2^n$ for the probability of $\mathcal{A}$, and for $i = 1, \dots, n$ write β_i for the influence of the ith variable on $\mathcal{A}$.

(i) State the basic identities and inequalities concerning the characteristic function $f = \mathbf{1}_{\mathcal{A}}$, its Fourier coefficients, the influences β_i, and the relevant consequence of the Bonami–Beckner inequality.

(ii) Show that if

$$\sum_{i=1}^{n} \beta_i^2 < \lambda^2/n$$

for some $\lambda > 0$ then

$$\sum_{A \neq \emptyset} |A| \alpha_A^2 < \lambda/4$$

and for $1 < p = 1 + \delta < 2$ we have

$$\sum_{A \neq \emptyset} |A| \delta^{|A|} \alpha_A^2 < \frac{1}{4} \lambda^{2/p} n^{1-2/p}.$$

(iii) Show that if n is sufficiently large then

$$\sum_{i=1}^{n} \beta_i^2 \geq t^2(1-t)^2(\log n)^2/n.$$

[Hint for Part (iii). Set

$$\lambda = t(1-t)\log n, \quad \gamma = 3(\log\log n)/\log n, \quad \delta = 1 - \gamma \quad \text{and} \quad p = 1 + \delta,$$

and make use of the inequalities in Part (ii). You may find it convenient to consider

$$\sum_{A \neq \emptyset} \alpha_A^2 = \sum_{1 \leq |A| < b} \alpha_A^2 + \sum_{|A| > b} \alpha_A^2,$$

where $b = (\log n)/3$.]

2 (i) State the Uniform Cover Inequality for projections of bodies, and deduce from it the Box Theorem.

(ii) Let $S_1, \dots, S_n$ be non-empty finite sets of integers. For $\emptyset \neq A \subset [n]$ put $S_A = \{\sum_{i \in A} s_i : s_i \in S_i \text{ for every } i \in A\}$, so that $S = S_{[n]}$ is the sum of all n sets and $S_{\{i\}} = S_i$ for every i. Show that there are constants $b_1, \dots, b_n > 0$ such that

$$|S| = \prod_1^n b_i \quad \text{and} \quad |S_A| \geq \prod_{i \in A} b_i \quad \text{for all} \quad A \subset [n].$$

(iii) Show also that if $|S_i| = 2$ for every i and $|S_{\{i,j\}}| = 4$ for all $1 \leq i < j \leq n$ then $|S| \geq \binom{n+1}{2} + 1$, and that this inequality is best possible for every n.

[Hint to Part (iii). Show that you may assume that $S_i = \{0, s_i\}$ and $0 < s_1 < \cdots < s_n$, and enumerate some elements of S that are guaranteed to be different: $0 < s_1 < s_2 < s_1 + s_2 < s_1 + s_3 < \dots$.]

3 (i) State the Balister–Bollobás Inequality and deduce from it the Madiman–Tetali Inequality.

(ii) Let G be a graph on $[n]$ with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. For $1 \leq i \leq n$, write b_i for the number of vertices $j < i$ that are joined to i. Show that G has at most

$$\prod_{i=1}^{n} (2^{b_i+1} - 1)^{1/d_i}$$

independent sets.

(iii) Give an infinite family of examples for which the bound above is attained.

4 (i) Use entropy methods to prove that if the complete graph K_n is the union of ℓ bipartite graphs $G_1, \ldots, G_\ell$, then $\sum_{i=1}^{\ell} |G_i| \geq n \log_2 n$, where $|G_i|$ is the order of G_i.

[Hint. Let $\chi_i : V(G_i) \to \{0, 1\}$ be a (proper) two-colouring of G_i. Let X be a vertex of K_n chosen uniformly at random, and for each i define a random variable Y_i by setting $Y_i = \chi_i(X)$ if $X \in V(G_i)$, and $Y_i = \chi_i(Z_i)$ if $X \notin V(G_i)$, where Z_i is a random vertex of $V(G_i)$, chosen uniformly and independently of all other choices. Note that $H(X|Y_1, \ldots, Y_\ell) = 0$.]

(ii) A weight $w(G)$ of a graph G is defined as follows. Let Z be a vertex of G chosen uniformly at random, and let $\widetilde{\chi}$ be a (proper) colouring of the vertices of G that minimizes the entropy $H(\widetilde{\chi}(Z))$. The *weight* of G is then $w(G) = |G|\ H(\widetilde{\chi}(Z))$. Show that if the complete graph K_n is the union of ℓ graphs $G_1, \ldots, G_\ell$ then $\sum_{i=1}^{\ell} w(G_i) \geq n \log_2 n$. Justify all the steps in your argument.

[Hint. Let X be a random vertex of K_n and define the random variable Y_i by setting $Y_i = \widetilde{\chi}_i(X)$ if $X \in V(G_i)$, and $Y_i = \widetilde{\chi}_i(Z_i)$ if $X \notin V(G_i)$, where Z_i is a random vertex of $V(G_i)$, chosen uniformly and independently of all other variables. Note that $H(X|Y_1, \ldots, Y_\ell) = 0$.]

(iii) Show that if $G = \bigcup_{i=1}^{\ell} G_i$ is a graph with n vertices and independence number at most α (i.e. G does not contain $\alpha + 1$ independent vertices) then $\sum_{i=1}^{\ell} w(G_i) \geq n \log_2(n/\alpha)$.

[Hint. Starting as in (ii), at most how large is the conditional entropy $H(X|Y_1, \ldots, Y_\ell)$?]

END OF PAPER