UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 9.00 to 12.00

PAPER 13

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Ω be a domain in $\mathbf{R}^n$.

(a) State and prove the mean value properties for a C^2 harmonic function in Ω.

(b) State and prove the strong maximum principle for a C^2 harmonic function in Ω.

(c) Prove that $u \in C^2(\Omega)$ is harmonic in Ω if and only if u is continuous in Ω and satisfies the following two conditions: (i) if $v \in C^2(\Omega)$ and $u - v$ has a local maximum at $x_0 \in \Omega$, then $\Delta\, v(x_0) \geqslant 0$ and (ii) if $v \in C^2(\Omega)$ and $u - v$ has a local minimum at $x_0 \in \Omega$, then $\Delta\, v(x_0) \leqslant 0$.

You may use without proof any standard existence theorem for harmonic functions provided you clearly state it.

2 For $1 \leqslant i, j \leqslant n$, let a_{ij} be bounded, measurable functions on a bounded domain $\Omega \subset \mathbf{R}^n$. Suppose that $a_{ij} = a_{ji}$ and $a_{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for some constant $\lambda > 0$ and all $x \in \Omega$, $\zeta \in \mathbf{R}^n$, and $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$D_i(a_{ij}D_ju) = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\,\Omega. \qquad (\star)$$

(a) Define what it means for a function $u \in W_0^{1,2}(\Omega)$ to be a weak solution to $(\star)$.

(b) Set $Q(u,v) = \int_\Omega a_{ij}D_iuD_jv$ for $u, v \in W_0^{1,2}(\Omega)$. Show that $(Q(u,u))^{1/2}$ defines a norm on $W_0^{1,2}(\Omega)$ equivalent to the usual norm, and use this fact to prove that the Dirichlet problem $(\star)$ has a unique weak solution $u \in W_0^{1,2}(\Omega)$.

(c) Find an appropriate functional $\mathcal{F} : W_0^{1,2}(\Omega) \to \mathbf{R}$ such that any critical point of $\mathcal{F}$ is a solution to the problem $(\star)$. Use $\mathcal{F}$ and the direct method of the calculus of variations to give another proof of existence of a weak solution to $(\star)$.

You may use without proof standard theorems in linear functional analysis and Sobolev space theory provided you clearly state them.

3 Let $n \geqslant 3$ and let Ω be a bounded domain in $\mathbf{R}^n$. Define a map $T : L^2(\Omega) \to W_0^{1,2}(\Omega)$ by setting $Tv = u$ where $u \in W_0^{1,2}(\Omega)$ solves $\Delta u = v$ weakly.

(a) Prove that the map T is well defined.

(b) Prove that T is a bounded, linear map.

(c) Suppose $w \in L^p(\Omega)$ for some $p > n$. Prove that the operator $K : L^2(\Omega) \to L^2(\Omega)$ defined by $Kv = wTv$ is compact.

(d) Deduce that if $w \in L^p(\Omega)$ for some $p > n$, then $\Delta u - wu = f$ is uniquely solvable in $W_0^{1,2}(\Omega)$ for each $f \in L^2(\Omega)$ if and only if $\Delta u - wu = 0$ has no non-trivial solutions in $W_0^{1,2}(\Omega)$.

You may use without proof standard theorems in linear functional analysis and Sobolev space theory provided you clearly state them.

4 Let $L\,u \equiv a_{ij}D_{ij}u + b_jD_ju$ be a uniformly elliptic operator in a bounded domain $\Omega \subset \mathbf{R}^n$, where the coefficients a_{ij}, b_j are bounded and measurable.

(a) State the weak maximum principle for $C^2(\Omega) \cap C^0(\overline{\Omega})$ subsolutions of the equation $L\,u = 0$ in Ω.

(b) Suppose that $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $f \in C^0(\overline{\Omega})$, u satisfies $Lu = f$ in Ω and v satisfies $Lv \geqslant 1$ in Ω and $v \leqslant 0$ on $\partial\,\Omega$. Prove that

$$u(x) \geqslant \left(\sup_{\Omega} f^+\right) v(x) + \inf_{\partial\,\Omega} u$$

for all $x \in \Omega$. Here $f^+(x) = \max\{f(x), 0\}$.

(c) Deduce that if Ω is a bounded domain in $\mathbf{R}^n$ with $0 \in \Omega$, then for any function $w \in C^2(\overline{\Omega})$,

$$w(x) \geqslant \tfrac{1}{2n}\left(\sup_{\Omega} (\Delta\,w)^+\right)(|x|^2 - d^2) + \inf_{\partial\,\Omega} w$$

for all $x \in \Omega$, where $d = \operatorname{diam}(\Omega)$.

5 (a) Prove that a function u is weakly differentiable in an open subset Ω of $\mathbf{R}^n$ if and only if it is weakly differentiable in a neighborhood of each point of Ω.

(b) Let $n \geqslant 2$ and let Ω be an open subset of $\mathbf{R}^n$, $x_0 \in \Omega$ and u a bounded function on Ω. If $u \in C^1(\Omega \setminus \{x_0\})$ with $Du \in L^1_{\mathrm{loc}}(\Omega)$, prove that u is weakly differentiable in Ω with the weak partial derivatives equal to the classical partial derivatives in $\Omega \setminus \{x_0\}$. Give an example to show that in case $n = 1$, this conclusion cannot be made under the same hypotheses on u.

(c) Let $\theta \in (0, 1]$. Prove that there exists a constant C depending only on n and θ such that

$$\int_{B_R} u^2 \leqslant CR^2 \int_{B_R} |Du|^2$$

for every function $u \in W^{1,2}(B_R)$ with $|\{x \in B_R : u(x) = 0\}| \geqslant \theta \omega_n R^n$, where B_R denotes an open ball in $\mathbf{R}^n$ with radius R. Here for a measurable subset A of $\mathbf{R}^n$, $|A|$ denotes the n-dimensional Lebesgue measure of A and $\omega_n = |B_1|$. (Hint: consider the case $R = 1$ first.)

6 (a) Let Ω be a domain in $\mathbf{R}^n$ and $u \in W^{1,2}(\Omega)$. For $h \neq 0$ and $k \in \{1, 2, \ldots, n\}$, let $\Delta_k^h u(x) = h^{-1}(u(x + he_k) - u(x))$ where e_k is the kth standard basis vector in $\mathbf{R}^n$. Prove that $\Delta_k^h u \in L^2(\Omega')$ and $\|\Delta_k^h u\|_{L^2(\Omega')} \leqslant \|Du\|_{L^2(\Omega)}$ for each subdomain $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$.

(b) Suppose $f \in L^2(\Omega)$ and $u \in W^{1,2}(\Omega)$ is a weak solution of

$$\Delta u = f \text{ in } \Omega.$$

Prove that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and that for each subdomain $\Omega' \subset\subset \Omega$, there exists a constant $C = C(n, \operatorname{dist}(\Omega', \partial\Omega))$ such that

$$\|u\|_{W^{2,2}(\Omega')} \leqslant C\left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$

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