UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2008 9.00 to 12.00

PAPER 12

COMPLEX DIFFERENTIAL EQUATIONS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A function $A : (z, w, \alpha) \mapsto A_\alpha(z, w)$ is analytic on the domain

$$\{(z, w, \alpha) \in \mathbb{C}^3 : |z| < R_1 \ , \quad |w| < S_1 \ , \quad |\alpha| < T_1\} \ .$$

Show that, for a suitable choice of $r, \varepsilon > 0$, the unique solution $f_\alpha : \mathbb{D}(0, r) \to \mathbb{C}$ of

$$f'_\alpha(z) = A_\alpha(z, f_\alpha(z)) \ ; \qquad f_\alpha(0) = 0 \tag{1}$$

exists and varies analytically with $\alpha \in \mathbb{D}(0, \varepsilon)$.

Show how to deduce that the solution of a differential equation

$$f'(z) = A(z, f(z)) \ ; \qquad f(0) = w_o$$

varies analytically with the initial value w_o.

Let $f_\alpha : D(0, r) \to \mathbb{C}$ be the solution of the differential equation (1) given above. Suppose that f_0 takes the value w_1 with multiplicity $k \geqslant 1$ at a point $z_1 \in \mathbb{D}(0, r)$. Prove or disprove the following statements.

(a) f_α takes the value w_1 in $\mathbb{D}(0, r)$ for $|\alpha| < \delta$ and δ sufficiently small.

(b) f_α takes the value w_1 with multiplicity k at some point of $\mathbb{D}(0, r)$ for $|\alpha| < \delta$ and δ sufficiently small.

2 Explain what it means to say that a linear differential equation

$$\boldsymbol{F}'(z) = \boldsymbol{A}(z)\boldsymbol{F}(z) \ ,$$

where $\boldsymbol{F}$ takes values in $\mathbb{C}^N$, has a *regular singularity.* What is the *residue of* $\boldsymbol{A}$ at such a singularity?

In the case where

$$\boldsymbol{A}(z) = \frac{\boldsymbol{R}}{z}$$

for a constant matrix $\boldsymbol{R}$ describe how the residue is related to the monodromy group of the differential equation.

Let $\boldsymbol{A}$ be an analytic, matrix-valued function on all of the Riemann sphere except for a finite number of regular singular points. Show that the sum of the residues at these points is $\boldsymbol{0}$ and identify $\boldsymbol{A}(z)$ in terms of these residues.

Let $\boldsymbol{B}, \boldsymbol{C}$ be two constant $N \times N$ matrices. Find the set S of singular points of the differential equation:

$$\boldsymbol{F}'(z) = \left(\frac{\boldsymbol{B} + z\boldsymbol{C}}{1 - z^3} \right) \boldsymbol{F}(z) \ .$$

Show that any linear differential equation on the entire Riemann sphere, that has all of its singular points regular and lying in S, must be of this form.

3 The differential equation

$$f'(z) = A(z, f(z)) \tag{3}$$

involves a rational function $A(z, w)$ and has no movable branch points in all of the Riemann sphere. Show that (3) is a Riccati equation.

Show that we can write any solution of a Riccati equation as

$$f(z) = \frac{f_1(z)}{f_2(z)}$$

where $\boldsymbol{F}(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ is an analytic solution of a suitable linear differential equation.

Let g, h, j be three solutions of the Riccati equation that take distinct values at a point z_o. Show that every solution of the differential equation can be written as a rational function of g, h and j.

4 Let (z_n) be a sequence of distinct points in $\mathbb{C} \setminus \{0\}$ that converge to ∞. Prove that there is an analytic function $f : \mathbb{C} \to \mathbb{C}$ that has zeros at the points (z_n) and nowhere else.

Denote the principal branch of the logarithm by Log. Prove that there are constants C_j with the following properties.

(a) If $|w| \leqslant \frac{1}{2}$, then
$$|\mathrm{Log}(1 - w)| \leqslant C_1 |w| \ .$$

(b) If $|w| \geqslant \frac{1}{2}$, then
$$\log(1 + |w|) \leqslant C_2 |w| \ .$$

(c) For all $w \in \mathbb{C}$,
$$|\mathrm{Log}(1 - w)| \leqslant C_3 |w| \ .$$

Deduce that, when the sequence (z_n) satisfies $\sum \frac{1}{|z_n|} < \infty$, then the infinite product

$$P(z) = \prod \left(1 - \frac{z}{z_n}\right)$$

is an analytic function satisfying

$$|P(z)| \leqslant \exp\left(C|z|\right)$$

for some constant C.

5 Let $f : \mathbb{C} \to \mathbb{P}$ be a meromorphic function with zeros (z_n) and poles (p_n) repeated according to their multiplicity. Let $f(0) \neq 0, \infty$. Define the *Nevanlinna characteristic of* f and prove Nevanlinna's First Theorem.

Deduce *Jensen's formula:*

$$\int_0^{2\pi} \log |f(Re^{i\theta})| \, \frac{d\theta}{2\pi} = \sum \left\{ \log \frac{R}{|z_n|} : |z_n| < R \right\} - \sum \left\{ \log \frac{R}{|p_n|} : |p_n| < R \right\} + \log |f(0)|$$

Show that, when $f : \mathbb{D} \to \mathbb{C}$ is a non-constant, bounded, analytic function, the zeros of f form a Blaschke sequence.

6 Explain what it means to say that a singularity of a differential equation is fixed or movable?

State and prove Painlevé's Determinateness theorem.

Determine the singularities of the differential equation

$$f'(z) = \frac{f(z)}{(z+1)(f(z)^2 - z^2)}$$

and determine whether they are fixed or movable.

END OF PAPER