UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 9.00 to 12.00

PAPER 11

INTRODUCTION TO FUNCTIONAL ANALYSIS

*Attempt not more than **THREE** questions, and not more than **TWO** from either section*

*There are **FIVE** questions in total*

The questions carry equal weight

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section I

1 What is a *filter*? What is a *convergent filter*? What is an *ultrafilter*?

State without proof a theorem which characterizes compact Hausdorff topological spaces in terms of ultrafilters.

Suppose that $\mathcal{U}$ is an ultrafilter on X and that $X = A_1 \cup \cdots \cup A_n$ is a partition of X into disjoint sets. Show that $A_j \in \mathcal{U}$ for some j. State and prove a converse result.

Define the image $f(\mathcal{F})$ of a filter on X under a mapping $f : X \to Y$. Show that if U is an ultrafilter then so is $f(\mathcal{U})$.

Let $\mathcal{U}$ be an ultrafilter on the natural numbers $\underset{\sim}{\mathrm{N}}$ which contains no finite sets. If $x = (x_n) \in l^\infty$, let $a_x(n) = (x_1 + \cdots + x_n)/n$. Show that $a_x(\mathcal{U})$ converges, to $l(x)$, say. Show that l is a positive linear functional of norm 1 on l^∞. Show that if $x_n \to l$ as $n \to \infty$, then $l(x) = l$. Show that if $x \in l^\infty$ and $S(x) = (x_2, x_3, \ldots)$ then $l(x) = l(S(x))$.

2 Suppose that (Ω, Σ, μ) is a measure space, with $\mu(\Omega) < \infty$, and that ν is a measure on Σ with $\nu(\Omega) < \infty$. Show that there exists a non-negative $f \in L^1(\mu)$ and a set $B \in \Sigma$ with $\mu(B) = 0$ such that $\nu(A) = \int_A f \, d\mu + \nu(A \cap B)$ for each $A \in \Sigma$.

What does it mean to say that ν is *absolutely continuous* with respect to μ? Use the result above to characterize such measures.

3 Suppose that G is a compact Hausdorff topological group. If $g \in G$ and $f \in C(G)$, let $l_g(f)(x) = f(g^{-1}(x))$. Show that this defines a continuous action (the left regular action) of G on $C(G)$.

Show further that the mapping $(g, \phi) \to l'_g(\phi)$ is a continuous action of G on the unit ball B' of $(C(G))'$, with the weak*-topology.

Explain briefly how this is used to establish the existence of left Haar measure μ_l on G.

By considering the right regular action and right Haar measure μ_r, show that μ_l is unique, and equal to μ_r.

Section II

4 Suppose that A is a unital Banach algebra. What is the *spectrum* $\sigma_A(a)$ of an element a of A? State, without proof, which subsets of the complex plane can be the spectrum of some element of some Banach algebra.

Show that if p is a polynomial, then $\sigma_A\left(p\left(a\right)\right) = p\left(\sigma_A\left(a\right)\right)$.

Suppose that B is a closed unital subalgebra of A, and that $b \in B$. Show that $\sigma_A(b) \subseteq \sigma_B(b)$, and the boundary of $\sigma_B(b)$ is contained in the boundary of $\sigma_A(b)$.

Suppose that $a \in A$. Show that there is a closed commutative unital subalgebra C of A for which $\sigma_C(a) = \sigma_A(a)$.

Suppose that $a \in A$. Let D be the smallest closed unital subalgebra of A containing a. Show that the complement of $\sigma_D(a)$ is connected.

5 Suppose that a and b are elements of a unital Banach algebra A. Show that if λ is a non-zero element of the spectrum of ab, then λ is in the spectrum of ba.

What is a *positive* element of a unital C^*-algebra A? Show that if $a \in A$ then a^*a is positive.

Suppose that a, b and $b^2 - a^2$ are positive. Show that $b - a$ is positive. Explain briefly why $b - a$ has a positive square root.

Suppose further that a is invertible. Show that b is also invertible.

END OF PAPER