UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 9.00 to 11.00

PAPER 10

TOPICS IN INFINITE GROUPS

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define what it means for a group G to have **max** and show that this is equivalent to every subgroup of G being finitely generated.

Show that the properties of being finitely generated and having max are preserved under extensions.

Let $d(H)$ be the minimum number of generators needed for the finitely generated group H.

The **Prüfer rank** of a group G (where G need not be finitely generated) is defined to be the supremum of $d(H)$ where H varies over the finitely generated subgroups of G (and is infinite if there is no supremum).

Give an example, with justification, of an infinitely generated group that has finite Prüfer rank.

Show that if G has finite Prüfer rank r then all subgroups and quotients of G have finite Prüfer rank at most r.

(You may use the fact that

$$\theta(\langle g_1, \ldots, g_k \rangle) = \langle \theta(g_1), \ldots, \theta(g_k) \rangle.)$$

Now suppose that N is a normal subgroup of G, with N having finite Prüfer rank r and G/N having finite Prüfer rank s.

Show that if H is a finitely generated subgroup of G with $H \cap N$ also finitely generated then $d(H) \leqslant r + s$.

Show further, by adapting your proof that finite generation is preserved by extensions or otherwise, that G has finite Prüfer rank at most $r + s$.

2 Define the free product $G_1 * G_2$ of two groups G_1 and G_2.

State and prove Klein's Combination Theorem.

Prove that the group generated by the Möbius transformations $f(z) = z + 2$ and $g(z) = z/(2z + 1)$ is free of rank 2.

If H has finite index i in the free group F_n of rank n then state and prove a formula for the rank of H.

(You may assume that any cover of a connected graph is a connected graph and that the fundamental group of a connected graph is free, with rank the number of edges in the complement of a maximal tree.)

For a finitely generated group G the quantity $d(G)$ is the minimum number of elements required to generate G. Show that $d(F_n) = n$.

The upper and lower rank gradients of a finitely generated group G are defined as the supremum and infimum respectively of $d(H)/[G : H]$ as H varies over the finite index subgroups of G.

What are the upper and lower rank gradients of F_n?

3 Assuming the universal property of free groups, define what it means for a group G to be finitely presented.

Show that if $G = N \rtimes_\phi H$ is the semidirect product of the finitely presented groups N and H then G is also finitely presented.

A group G is said to be poly-$\mathbb{Z}$ if there exist subgroups $G_0, G_1, \ldots, G_n$ with

$$I = G_n \trianglelefteq G_{n-1} \trianglelefteq \ldots \trianglelefteq G_0 = G$$

such that each successive quotient G_i/G_{i+1} is isomorphic to the integers $\mathbb{Z}$.

Show that the property of being poly-$\mathbb{Z}$ is preserved by subgroups and extensions.

Show that a poly-$\mathbb{Z}$ group is finitely presented.

Show that there exists a finitely presented group which is not poly-$\mathbb{Z}$ but such that G has subgroups $G_0 = G, G_1, \ldots, G_n = I$ as above, but where each successive quotient G_i/G_{i+1} is now a torsion free abelian group.

4 Show that if G is a finitely generated group and n is any strictly positive integer, there are only finitely many subgroups of index n in G.

Define the finite residual R_G of a group G and say what it means for G to be residually finite.

Define what it means for a group to be Hopfian and prove that a finitely generated residually finite group is Hopfian. If $\theta : G \to G$ is a surjective homomorphism then what is the relationship between the kernel of θ and R_G?

Now let G be the Baumslag-Solitar group with finite presentation

$$\langle a, t | ta^2t^{-1} = a^3 \rangle.$$

Give a set of generators for the commutator subgroup G'.

Show that there exists a surjective homomorphism θ that is not injective. By repeatedly composing θ or otherwise, show that the second commutator subgroup G'' is contained in the finite residual R_G.

(You may assume standard facts about HNN extensions.)

5 Describe the construction of a finitely generated, infinite, residually finite group G such that all elements have finite order.

Is it possible for there to be a surjective homomorphism from G to an infinite soluble group?

END OF PAPER