UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2008 9.00 to 12.00

PAPER 1

VON NEUMANN ALGEBRAS

*Attempt no more than **THREE** questions.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(a)** State and prove the spectral theorem for compact self–adjoint operators on a Hilbert space.

(b) Define the trace–class norm on the space of finite–rank operators. Prove that it is a norm and that the dual space can be identified with $B(H)$.

(c) Prove that the Hilbert–Schmidt operators on a Hilbert space H themselves form a Hilbert space. Hence or otherwise deduce that every unitary representation of a compact metric group on H can be written as an orthogonal direct sum of finite–dimensional irreducible invariant subspaces. [You may assume any properties of Haar measure that you require.]

2 **(a)** Define what is meant by a *Hadamard space*.

(b) Show that there is a unique geodesic between any two points in a Hadamard space.

(c) If x, y, a are three distinct points in a Hadamard space (X, d) and x_s and y_t are points on the geodesic segments $[a, x]$ and $[a, y]$ such that $d(a, x_s) = s \cdot d(a, x)$ and $d(a, y_t) = t \cdot d(a, y)$, prove that $d(x_s, y_t)$ is no greater than the corresponding distance in the triangle in the euclidean plane with sides $d(a, x)$, $d(a, y)$ and $d(y, z)$.

(d) Prove that every closed ball in a Hadamard space is geodesically convex, i.e. contains all geodesics connecting any two of its points.

3 **(a)** State and prove Stone's theorem for unitary representations of $\mathbb{T}$.

(b) State and prove Stone's theorem for unitary representations of $\mathbb{R}$. [You may assume any properties of the Borel functional calculus that you require.]

4 Explain carefully what it means for a countable group Γ to be amenable. State *without proof* an equivalent condition in terms of left invariant states on *–subalgebras of $\ell^\infty(\Gamma)$.

Prove that the following properties are equivalent:

(1) Γ is amenable;

(2) there is a a sequence of probability measures μ_n on Γ such that $\|\lambda(g)\mu_n - \mu_n\|_1 \to 0$ for each $g \in \Gamma$;

(3) there is a sequence of finite subsets F_n of Γ such that $|F_n \triangle gF_n|/|F_n| \to 0$ for each $g \in \Gamma$.

5 **(a)** Show that if a group of *affine* isometries of a Hilbert space H leaves invariant a closed bounded convex subspace C of H, it must have a fixed point in C.

(b) Let E be a separable Banach space. Prove that the closed unit ball E_1^* of its dual E^* is metrisable and compact in the weak topology.

(c) Let S be a bounded operator on E inducing a bounded operator T on E^*. Suppose that T leaves invariant a weakly closed convex subset C of E_1^*. Prove that T has a fixed point in C.

(d) Let U be a unitary operator on a Hilbert space. let $T_n = n^{-1}(I + U + \cdots + U^{n-1})$ and let P be the orthogonal projection onto the fixed points of U. Prove that $T_n \to P$ in the strong operator topology.

6 State and prove von Neumann's double commutant theorem and Kaplansky's density theorem for unital *–subalgebras of $B(H)$.

7 **(a)** Let V be a finite–dimensional complex vector space, $A \subset \mathrm{End}\, V$ a unital *–subalgebra and Ω a vector in V such that $A\Omega = V = A'\Omega$, where A' is the commutant of A on V. Define the modular operators J and Δ and prove that $JAJ = A'$.

(b) Prove that if Γ is a countable group and λ is the left regular representation on $\ell^2(\Gamma)$, then the von Neumann algebra $\lambda(\Gamma)''$ is a factor if and only if every non–identity conjugacy class of Γ is infinite.

(c) Let $M \subset B(H)$ be a von Neumann algebra and Ω a vector in H such that $M\Omega$ and $M'\Omega$ are dense in H and $(ab\Omega, \Omega) = (ba\Omega, \Omega)$ for $a, b \in M$. Define the modular operator J and prove that $JMJ = M'$. Hence or otherwise, identify the commutant of $\lambda(\Gamma)$ in (b).

8 **(a)** What does it mean for a bounded self–adjoint operator to be positive? Prove that a positive operator has a unique positive square root.

(b) State and prove a theorem on the polar decomposition of a bounded operator on a Hilbert space.

(c) Let $M \subset B(H)$ be a von Neumann algebra and H_1 and H_2 be two closed subspaces of H invariant under M. Prove that $\overline{H_1 + H_2} \cap H_2^{\perp}$ and $H_1 \cap (H_1 \cap H_2)^{\perp}$ are unitarily equivalent as M–modules.

(d) Prove that H_1 and H_2 are unitarily equivalent if and only if each is unitarily equivalent to a closed invariant subspace of the other.

END OF PAPER