UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 12.00

PAPER 9

ISOPERIMETRY AND CONCENTRATION OF MEASURE

Answer ***THREE*** *questions*

There are ***FIVE*** *questions*

The questions carry equal weight

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe briefly the properties of the set I_X of isometries of a compact metric space (X, d). What does it mean to say that I_X acts *transitively* on X?

Suppose that I_X acts transitively on (X, d). Show that there is a unique Borel probability measure μ on X such that $\int_X f(x)\, d\mu(x) = \int_X f(g(x))\, d\mu(x)$ for every $g \in I_X$ and $f \in C(X)$.

[You may assume that the set $P(X)$ of Borel probability measures on X is a compact metrizable subset of $C(X)^*$ under the weak*-topology. You may assume Hall's marriage theorem: if so, you should state it clearly.]

2 What does it mean to say that a random variable X is sub-Gaussian, with exponent b? Show that a bounded random variable X is sub-Gaussian if and only if $\mathbf{E}(X) = 0$.

Suppose that X is sub-Gaussian, with exponent b. Show that $\mathbf{P}(|X| > R) \leqslant 2e^{-R^2/2b^2}$. Show further that $\|X\|_{2k} \leqslant b\sqrt{2k}$, for $k \geqslant 2$. Show that $\|X\|_2^3 \leqslant 4b^2 \|X\|_1$.

[If you use Littlewood's inequality, you should prove it.]

3 Suppose that $\mathcal{E}$ is the ellipsoid of maximum volume contained in the unit ball B_E of a d-dimensional normed space $(E, \|.\|_E)$, and that $|\,.\,|$ is the inner-product norm with unit ball $\mathcal{E}$. State inequalities relating $\|.\|_E$ and $|\,.\,|$. Show that there exists a $|\,.\,|$-orthonormal basis $(e_1, \ldots, e_d)$ with $\|e_i\|_E \geqslant 1/4$ for $1 \leqslant i \leqslant d/2$.

Use this orthonormal basis to identify E with $\mathbf{R}^d$, and let γ_d be normalized Gaussian measure on $\mathbf{R}^d$. Show that there is a positive constant c, which does not depend on d or $\|.\|_E$, such that

$$\int_{\mathbf{R}^d} \|x\|_E \; d\gamma_d(x) \geqslant c\sqrt{\log d}.$$

4 Define the *volume ratio* $vr(E)$ of a finite-dimensional normed space $(E, \|.\|_E)$. Show that there exists a constant C, independent of k, such that $vr(l_1^{2k}) \leqslant C$.

Suppose that E has dimension $2k$. Let $\mathcal{E}$ be the ellipsoid of maximum volume contained in the unit ball B_E of $(E, \|.\|_E)$, and let $|\,.\,|$ be the inner-product norm with unit ball $\mathcal{E}$. Show that there exists a constant L, independent of k and $\|.\|_E$, such that there exist two k-dimensional subspaces of E, orthogonal with respect to the inner product, on each of which

$$\|x\|_E \leqslant |x| \leqslant L\,\|x\|_E\,.$$

[You should establish any results about ϵ-nets that you need. The Euclidean ball in $\mathbf{R}^{2k}$ has volume $\pi^k/k!$.]

5 Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are probability measures on a compact metric space (X, d). Show that the following quantities are equal:

(i) $m_d(\mathbf{P}, \mathbf{Q}) = \sup\{\int_X f\, d\mathbf{P} + \int_X g\, d\mathbf{Q} : f, g \in C(X), f(x) + g(y) \leqslant d(x, y)\}$;

(ii) $W(\mathbf{P}, \mathbf{Q}) = \inf\{\int_{X \times X} d(x, y)\, d\pi(x, y) : \pi \in P(X, Y)$ with marginals $\mathbf{P}$ and $\mathbf{Q}\}$.

Show also that they are equal to the quantities

(i) $m_L(\mathbf{P}, \mathbf{Q}) = \sup\{\int_X f\, d\mathbf{P} + \int_X g\, d\mathbf{Q} : f, g \in Lip(X) : f(x) + g(y) \leqslant d(x, y)\}$;

(iv) $\gamma(\mathbf{P}, \mathbf{Q}) = \sup\{|\int_X f\, d\mathbf{P} - \int_X f\, d\mathbf{Q}| : f \in Lip(X), \|f\|_L \leqslant 1\}$.

END OF PAPER