UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2007 9.00 to 12.00

PAPER 89

ARITHMETIC GEOMETRY

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a scheme. What does it mean to say that X is *integral*? Show that $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

Let X be a scheme whose connected components are open. Show if $e \in \Gamma(X, \mathcal{O}_X)$ is an idempotent (i.e. $e^2 = e$) then $V((1-e))$ is an open and closed subset of X. Deduce that there is a bijection between the set of connected components of X, and the set of idempotents in $\Gamma(X, \mathcal{O}_X)$ which are indecomposable (i.e. cannot be written as the sum of two non-zero idempotents).

2 Let X be a scheme of characteristic $p > 0$. Define the *Frobenius morphism* $F_X : X \to X$. Show that F_X is an isomorphism if and only if all the local rings $\mathcal{O}_{X,x}$ are perfect.

Let $f : X \to Y$ be a finite surjective morphism of normal schemes of characteristic p. Identify the function field $k(Y)$ with a subfield of $k(X)$ via f^*. Show that

(i) if $k(X)^p \subset k(Y)$ then there is a unique morphism $g : Y \to X$ with $F_X = g \circ f$;

(ii) if $k(X)^p \supset k(Y)$ then there is a unique morphism $h : X \to Y$ such that $f = h \circ F_X$.

3 What does it mean to say that a morphism of schemes is *flat*? Show that an open immersion is always flat, and that a closed immersion between connected schemes is flat if and only if it is an isomorphism.

Let $f : X \to Y$ be a finite and flat morphism of irreducible schemes. For a point $y \in Y$ let $X_y = X \times_Y \operatorname{Spec} k(y)$ denote the fibre of f above y. Show that the function $y \mapsto \dim_{k(y)} \Gamma(X_y, \mathcal{O}_{X_y})$ is constant.

Hence show that if C is a curve (integral scheme of dimension one) over a field and $\pi : C' \to C$ is its normalisation, then π is flat if and only if $C' = C$.

4 Let R be a discrete valuation ring with uniformiser π and field of fractions F. Let X be the closed subscheme of $\mathbb{P}^3_R = \operatorname{Proj} R[x_0, x_1, x_2, x_3]$ defined by the ideal

$$(x_1^2 - \pi^2 x_0 x_2, x_1 x_3 - \pi x_2^2, x_1 x_2 - \pi x_0 x_3, x_0 x_3^2 - x_2^3)$$

(i) Show that there is an isomorphism $\mathbb{P}^1_F \simeq X \otimes_R F$ given by $(y_0, y_1) \mapsto (y_0^3, \pi y_0^2 y_1, y_0 y_1^2, y_1^3)$.

(ii) Show that the open subscheme $\{x_3 \neq 0\} \subset X$ is isomorphic to $\mathbb{A}^1_R$. Show also that if $U \subset X$ is the open subscheme $\{x_0 \neq 0\}$ then $U = \operatorname{Spec} B$ is affine and B is a torsion-free R-module. Deduce that X is proper and flat over $\operatorname{Spec} R$.

(iii) Let $s \in \operatorname{Spec} R$ be the closed point. Show that the base change map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_s, \mathcal{O}_{X_s})$ is not an isomorphism.

5 Let $\mathcal{F}$ be a quasi-coherent sheaf on a separated scheme X. Define the *Čech cohomology* groups $H^p(X, \mathcal{F})$ of $\mathcal{F}$ on X. Explain why, for any exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, there is a long exact sequence of cohomology

$$H^p(X, \mathcal{F}') \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}'') \to H^{p+1}(X, \mathcal{F}') \to \dots$$

Let k be a field and X the complement of the origin in $\mathbb{A}^2_k$. Show that $H^1(X, \mathcal{O}_X)$ is infinite-dimensional.

END OF PAPER