UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 1.30 to 4.30

PAPER 88

ASYMPTOTIC STRUCTURE AND QUASIRANDOMNESS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A be a set of integers, let C be a constant, and suppose that $|A+A| \leqslant C|A|$. Prove that $|2A-2A| \leqslant C^4|A|$. [*You may assume Menger's theorem.*]

2

(i) State and prove Roth's theorem on arithmetic progressions with a bound of the form $CN/\log\log N$.

(ii) Suppose that A is a set of integers such that $|2A-2A| \leqslant C|A|$, and let $N > C|A|$ be a prime. Prove that A has a subset A' of size at least $|A|/2$ that is 2-isomorphic to a subset of $\mathbb{Z}_N$.

(iii) Deduce that if $|A|$ is sufficiently large then A must contain an arithmetic progression of length 3. [*You may assume that there is a prime between $C|A|$ and $2C|A|$.*]

3

(i) State and prove Szemerédi's regularity lemma.

(ii) Explain how the regularity lemma can be used to give a proof of Roth's theorem. [*You may assume a suitable counting lemma for tripartite graphs.*]

4

(i) What does it mean to say that a tripartite 3-uniform hypergraph is α-quasirandom?

(ii) Let H be a quadripartite 3-uniform hypergraph with vertex sets X, Y, Z and W. Let the densities of the subhypergraphs $H(X,Y,Z)$, $H(X,Y,W)$, $H(X,Z,W)$ and $H(Y,Z,W)$ be p, q, r and s, respectively, and suppose that these subhypergraphs are all α-quasirandom. Prove that the number of simplices in H differs from $pqrs|X||Y||Z||W|$ by at most $C\alpha^{1/8}|X||Y||Z||W|$ for some absolute constant C.

(iii) Can you generalize the result of (ii) to prove a counting lemma for 3-uniform hypergraphs — that is, deal with more than just simplices?

END OF PAPER