UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 12.00

PAPER 87

LAMBDA–CALCULUS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

For all $m < n$, results which you have proved in answering question m (or which would have been proved if you had attempted question m) may be assumed in your answer to question n.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by a λ-term, and by β-equivalence of two λ-terms. Prove the Church–Rosser theorem, that two λ-terms are β-equivalent if and only if they have a common β-reduct. Deduce that an arbitrary λ-term is β-equivalent to at most one term in normal form. Give an example of a λ-term having no normal form.

2 Explain what is meant by a CL-term (or combinator term) in a given set X of free variables, and by the statement that two CL-terms are weakly equivalent. Show how to associate with each λ-term M a CL-term M^*, with the same set of free variables, in such a way that $x^* = x$ for each variable x, $(MN)^* = M^*N^*$ for all M and N, and M and N are β-equivalent whenever M^* and N^* are weakly equivalent. By considering the λ-terms $\lambda x.x$ and $\lambda x.((\lambda y.y)x)$, or otherwise, show that the converse of the last statement fails.

[You may assume the Church–Rosser property for weak equivalence of CL-terms.]

3 (*a*) Explain what is meant by a c.p.o. (with least element) and by a continuous map of c.p.o.'s. If D and D' are c.p.o.'s, show that the set $[D \to D']$ of continuous maps $D \to D'$ can be given the structure of a c.p.o. Deduce that the category of c.p.o.'s and continuous maps between them is cartesian closed.

(*b*) Let $\mathcal{C}$ be a cartesian closed category, and D an object of $\mathcal{C}$ satisfying $D^D \cong D$. Explain briefly how the set of morphisms $D \to D$ in $\mathcal{C}$ may be made into a model of the (untyped) λ-calculus satisfying the β- and η-rules (i.e., such that $\beta\eta$-equivalent terms have the same interpretation).

4 Explain what is meant by an embedding–projection pair of continuous maps between c.p.o.'s. Given a c.p.o. D_0, define $D_n = [D_{n-1} \to D_{n-1}]$ for $n > 0$; show that if we are given an embedding–projection pair $(\phi_0 \colon D_0 \to D_1, \psi_0 \colon D_1 \to D_0)$, we may obtain such pairs $(\phi_n \colon D_n \to D_{n+1}, \psi_n \colon D_{n+1} \to D_n)$ for all n by setting

$$\phi_n(f) = \phi_{n-1} \circ f \circ \psi_{n-1} \quad , \quad \psi_n(g) = \psi_{n-1} \circ g \circ \phi_{n-1} \ .$$

Explain briefly how this result may be used to construct a c.p.o. D_∞ satisfying $D_\infty \cong [D_\infty \to D_\infty]$, equipped with embedding–projection pairs $(D_n \to D_\infty, D_\infty \to D_n)$ for all n.

5 Let $\mathcal{C}$ be a cartesian closed category having just two objects 1 (the terminal object) and D, and suppose $\mathcal{C}$ is not a preorder. Show that we necessarily have $D \times D = D^D = D$, and that the monoid M of morphisms $D \to D$ in $\mathcal{C}$ comes equipped with distinguished elements π, π' and ϵ, a unary operation $(-)^*$ and an additional binary operation $\langle -, - \rangle$ satisfying

$$\pi\langle x, y\rangle = x \quad , \quad \pi'\langle x, y\rangle = y \quad , \quad \langle \pi z, \pi' z\rangle = z \quad ,$$

$$\epsilon\langle x^*\pi, \pi'\rangle = x \quad \text{and} \quad (\epsilon\langle y\pi, \pi'\rangle)^* = y$$

for all $x, y, z \in M$. Conversely, given a monoid M with this additional structure, show that the element $(\pi')^*$ is idempotent, and hence construct a two-object cartesian closed category such that M appears as the monoid of endomorphisms of its non-terminal object.

END OF PAPER