UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 1.30 to 3.30

PAPER 86

IMAGING, BOUNDARY VALUE PROBLEMS AND INTEGRABILITY

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) By assuming the validity of the Fourier transform, prove the validity of the following transform pair:

$$\hat{q}(k) = \int_0^\infty e^{-ikx} q(x)dx,$$

$$q(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{q}(k)dk + \tfrac{c}{2\pi} \int_L e^{ikx} \hat{q}(-k + i\alpha)dk,$$

where α is a positive constant, c is a constant and the contour L is defined by

$$k_R^2 - k_I^2 + \alpha k_I = 0, \quad k_I > 0,$$

with the orientation shown in Figure 1.1.

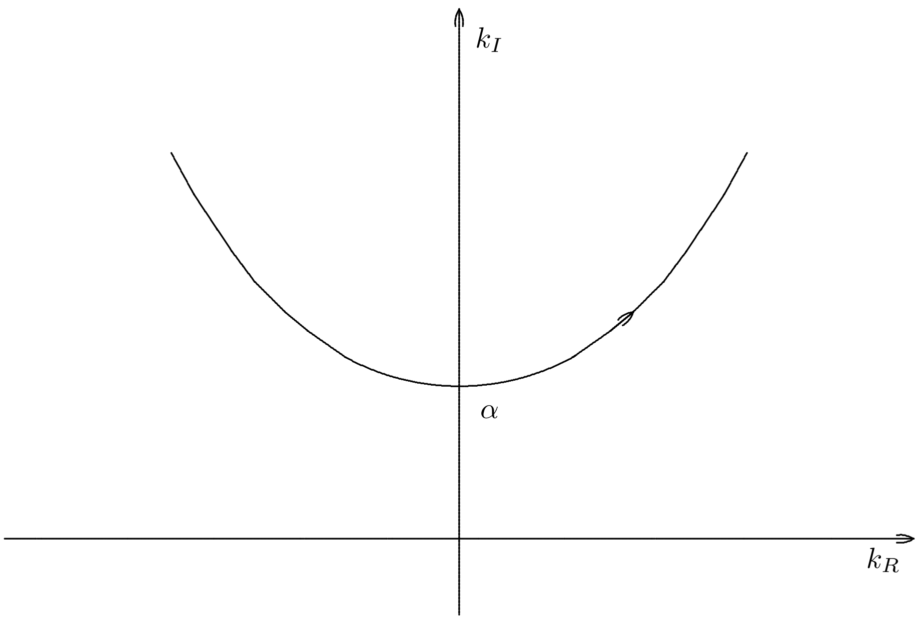

Figure 1.1

(b) Let $q(x,t)$ solve any of the following two initial-boundary value problems:

$$q_t = q_{xx} + \alpha q_x, \quad 0 < x < \infty, \quad 0 < t < T,$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty,$$

$$q(0,t) = g_0(t), \quad or \quad q_x(0,t) = g_1(t), \quad 0 < t < T,$$

where T is a positive constant, $\{g_j(t)\}_0^1$ and $q_0(x)$ are smooth functions, $q_0(x)$ has sufficient decay as $x \to \infty$, and $q_0(0) = g_0(0)$, or $\dot{q}_0 = g_1(0)$. For each of these problems use the transform pair defined in (a) to express $q(x,t)$ as an integral in the complex k-plane.

(c) In the case of the Dirichlet boundary value problem, prove that $q(x,t)$ solves the PDE, and that it also satisfies

$$q(x,0) = q_0(x), \quad 0 < x < \infty,$$

$$q(0,t) = g_0(t), \quad 0 < t < T.$$

2

(a) Let $W(z, \bar{z}, k)$ be defined by

$$W = e^{-ikz - \frac{\lambda}{ik}\bar{z}} \left[(q_z + ikq)dz - (q_{\bar{z}} + \tfrac{\lambda}{ik} q) d\bar{z} \right], \quad k \in \mathbb{C} \backslash \{0\}, \tag{1}$$

where λ is a constant, $z = x + iy$ and bar denotes complex conjugation.

Show that W is closed iff q satisfies the elliptic PDE

$$q_{z\bar{z}} - \lambda q = 0, \tag{2}$$

and derive the associated global relations.

(b) Assume that there exists a function $q(z, \bar{z})$, z in the first quadrant of the complex z-plane, with sufficient smoothness and decay, which satisfies (2) with $\lambda > 0$. By letting $W = d(\mu(z, \bar{z}, k) \exp[-ikz - \lambda\bar{z}/ik])$ and by performing the spectral analysis of the differential form W, find an integral representation for $q(z, \bar{z})$ in the complex k-plane.

(c) Let the function $q(x,y)$ solve equation (2) with $\lambda > 0$ in the first quadrant of the complex z-plane, with Dirichlet boundary conditions,

$$q(0,y) = g_1(y), \quad 0 < y < \infty,$$

$$q(x,0) = g_2(x), \quad 0 < x < \infty,$$

where the functions $\{g_j\}_1^2$ have sufficient smoothness and decay and satisfy $g_1(0) = g_2(0)$.

Use the integral representation derived in (b), as well as the associated global relations to express $q(x,y)$ as an integral in the complex k-plane.

3 Let the scalar function $\mu(x_1, x_2, k)$ satisfy

$$\tfrac{1}{2}\left(k+\tfrac{1}{k}\right)\tfrac{\partial\mu}{\partial x_1}+\tfrac{1}{2i}\left(k-\tfrac{1}{k}\right)\tfrac{\partial\mu}{\partial x_2}+f(x_1,x_2)\mu=g(x_1,x_2), \tag{1}$$
$$-\infty < x_1, x_2 < \infty, \quad k \in \mathbb{C}\,,$$

where f and g have sufficient smoothness and also they decay as $|x_1|^2+|x_2|^2 \to \infty$.

(a) Use an appropriate change of variables to reduce equation (1) to the equation

$$\nu(|k|)\tfrac{\partial\mu}{\partial\bar{z}}+f(x_1,x_2)\mu=g(x_1,x_2), \tag{2}$$

where $\nu(|k|)$ and z are to be determined.

(b) Assume that the solution $M(x_1, x_2, k)$ of the equation

$$\nu(|k|)\tfrac{\partial M}{\partial\bar{z}}=f(x_1,x_2),$$
$$M=0\left(\tfrac{1}{z}\right), \quad z\to\infty,$$

satisfies

$$M(x_1,x_2,k^{\pm})=\mp P^{\mp}\hat{f}(\rho,\theta)-\int_{\tau}^{\infty} f(\tau'\cos\theta-\rho\sin\theta,\tau'\sin\theta+\rho\cos\theta)d\tau',$$

where

$$k^{\pm}=\lim_{\varepsilon\to 0}(1\mp\varepsilon)e^{i\theta}, \quad 0\leqslant\theta\leqslant 2\pi, \quad \varepsilon>0,$$

$P^{\pm}$ denote the usual projectors in the variable ρ, $\hat{f}$ denotes the Radon transform of f, and the local coordinates (ρ, τ, θ) are indicated in Figure 3.1.

By employing equation (2), derive the Attenuated Radon transform pair, which provides the mathematical basis of Single Photon Emission Computerized Tomography.

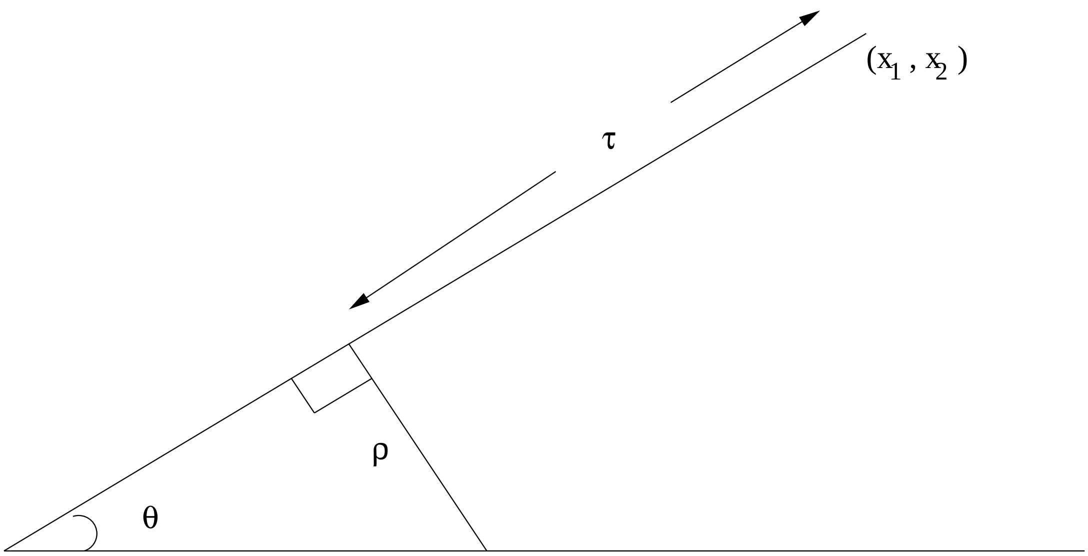

Figure 3.1

END OF PAPER