UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 1.30 to 4.30

PAPER 85

SYMMETRIC DYNAMICAL SYSTEMS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the fourth-order set of ODEs

$$\begin{aligned} \ddot{u} + \lambda u &= \kappa \dot{u} - uv + suw, \\ \dot{v} &= -v + u^2, \\ \dot{w} &= -\tau w + \frac{1}{\tau} u^2, \end{aligned} \tag{1}$$

where λ, κ are parameters and s and τ are positive constants.

(a) Carry out an extended centre manifold reduction of (1) near the codimension-two point $\lambda = \kappa = 0$. You need only compute the quadratic order approximation to the centre manifold. Explain carefully why near $\lambda = \kappa = 0$ the dynamics are qualitatively well described by the second-order ODE

$$\ddot{u} + \lambda u = \kappa \dot{u} + \left(\frac{s}{\tau^2} - 1 \right) u^3 + 2 \left(1 - \frac{s}{\tau^3} \right) u^2 \dot{u}. \tag{2}$$

Hint: consider the rescaling $(u, \kappa, d/dt) = \varepsilon(\tilde{u}, \tilde{\kappa}, d/d\tilde{t})$, $\lambda = \varepsilon^2 \tilde{\lambda}$.

(b) Now set $\tau = 1$ and investigate local bifurcations in (2) in the two cases (i) $s \gg 1$ and (ii) $s \ll 1$. Sketch phase portraits in the (κ, λ) plane on both sides of each bifurcation line. Discuss the existence of periodic orbits, applying Dulac's criterion.

(c) Without further calculation, for both cases (i) and (ii), describe possible additional (global) bifurcations that would complete the division of the (κ, λ) plane into regions of distinct dynamics.

2 Consider the 1D Lorenz map

$$x_{n+1} = f_L(x_n) \equiv \mathrm{sgn}(x_n)(-\mu + x_n^2)$$

where we define $\mathrm{sgn}(x) = x/|x|$ if $x \neq 0$, $\mathrm{sgn}(0) = 1$, and μ is a parameter. We say that a 1D map $f(x, \mu)$ has a *global bifurcation* at $\mu = \mu_0$ if $f(x, \mu_0)$ is continuous, $f(0, \mu_0) = 0$ and $f(x, \mu)$ is discontinuous for $0 < |\mu - \mu_0| < \varepsilon$ for some $\varepsilon > 0$.

(a) Sketch $f_L(x)$ and $g(x) \equiv -\mathrm{sgn}(x) f_L(x)$ for μ small, and both positive and negative. Hence describe the orbits created or destroyed in the global bifurcation at $\mu = 0$.

(b) Using induction, or otherwise, show that for any x

$$f_L^n(x) = (-1)^n g^n(x) \prod_{k=0}^{n-1} \mathrm{sgn}\left(g^k(x)\right), \tag{3}$$

where $f^n(x)$ denotes the n^{th} iterate of x under the map $f(x)$, i.e. repeated composition.

(c) Now let $\{x_1, \ldots, x_n\}$ be a period–n orbit of $g(x)$. Using your sketch from part (a) show that

$$\mathrm{sgn}\left(\tfrac{d}{dx} g^n(\hat{x})\right) = (-1)^n \prod_{k=1}^{n} \mathrm{sgn}(x_k), \tag{4}$$

where $\hat{x}$ is any one of the points $\{x_1, \ldots, x_n\}$.

(d) Combining (3) and (4), show that if $\frac{d}{dx} g^n(\hat{x}) > 0$ then $\hat{x}$ lies on a period–n orbit of $f_L(x)$, while if $\frac{d}{dx} g^n(\hat{x}) < 0$ then $\hat{x}$ lies on a period–$2n$ orbit of $f_L(x)$.

(e) Identify the bifurcation that occurs in $g(x)$ when $\mu = \frac{3}{4}$. Locate a stable period-2 orbit of $g(x)$ when $\mu = 1$ and hence show graphically that $f_L^2(x)$ undergoes a global bifurcation at $\mu = 1$. Describe, with justification, the orbits that are created or destroyed in this global bifurcation.

3 (a) Briefly outline the theory of steady-state local bifurcations with symmetry. You should include definitions of the terms

absolute irreducibility,

isotropy subgroup,

fixed point subspace,

and the *normaliser* $N(\Sigma)$ of an isotropy subgroup Σ.

State the Equivariant Branching Lemma and discuss the role of fixed point subspaces in understanding the dynamics. Theorems or lemmas should be stated carefully but need not be proved.

(b) Describe the generic steady-state local bifurcations with dihedral group D_n symmetry in the cases $n = 3, 4$ and 5, where in every case D_n acts on $\mathbb{R}^2 \simeq \mathbb{C}$ by its natural 2D representation.

4 (a) Let $\mathcal{G}$ be a finite group acting on $\mathbb{R}^n$. What does it mean to say that $\Sigma \subseteq \mathcal{G}\times S^1$ is a $\mathbb{C}$-*axial isotropy subgroup*?

(b) State the Equivariant Hopf Theorem for bifurcations in the presence of a finite group $\mathcal{G}$ acting by 2 copies of the same absolutely irreducible representation on $V \oplus V \simeq \mathbb{R}^n$.

(c) Consider Hopf bifurcation with $\mathcal{G} = D_4$ symmetry, where the action of $\mathcal{G}\times S^1$ on $\mathbb{R}^4 \simeq \mathbb{C} \oplus \mathbb{C}$ is generated by

$$\rho(w_1, w_2) = (\mathrm{i}w_1, \mathrm{i}w_2),$$
$$m_x(w_1, w_2) = (\overline{w}_1, \overline{w}_2),$$

$$\tau_\phi \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Find three distinct branches of periodic orbits that can be guaranteed generically to appear at such a Hopf bifurcation, making use of the coordinate transformation

$$z_1 = \frac{1}{2}\overline{(w_1 + \mathrm{i}w_2)}, \qquad z_2 = \frac{1}{2}(w_1 - \mathrm{i}w_2).$$

(d) Prove that the relevant amplitude equations, up to and including cubic order terms, take the form

$$\dot{z}_1 = (\mu + \mathrm{i}\omega)z_1 - az_1|z_1|^2 - bz_1|z_2|^2 - c\overline{z}_1 z_2^2,$$
$$\dot{z}_2 = (\mu + \mathrm{i}\omega)z_2 - az_2|z_2|^2 - bz_2|z_1|^2 - c\overline{z}_2 z_1^2.$$

Hence demonstrate that periodic orbits on the three distinct branches generically have different amplitudes.

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