UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 9.00 to 11.00

## PAPER 84

## QUANTUM FLUIDS

*Attempt **TWO** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** The energy functional of a quantum fluid described by a wavefunction $\psi$ is given by

$$E = \int \left[ \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{V_0}{2} |\psi|^4 + \frac{W_0}{3} |\psi|^6 \right] d\mathbf{x},$$

where $V_0$ and $W_0$ are the effective two- and three-body interaction potentials.

(i) Write down the corresponding (generalised) Gross-Pitaevskii equation, given by

$$i\hbar\psi_t = \delta E / \delta \psi^*.$$

(ii) Write down the stationary equation for the equilibrium state of the fluid for a given number of particles.

(iii) Relate the chemical potential, $\mu$, to the uniform number density, $n_0 = |\psi_0|^2$ of the ground state $\psi_0$ =const.

(iv) Show that the dimensionless form of the stationary equation you obtained in (ii) can be written as

$$\nabla^2 \tilde{\psi} + (1 - \alpha |\tilde{\psi}|^2 - \beta |\tilde{\psi}|^4) \tilde{\psi} = 0, \qquad \tilde{\psi} \to 1 \quad \text{at} \quad \text{infinity}, \qquad (*)$$

where $\alpha$ and $\beta$ are constants. Specify $\alpha$, $\beta$ and the unit of distance.

(v) Use $(*)$ to write down the equation for the amplitude $R(r)$ of the straight-line vortex of winding number $\mathcal{N}$, that is positioned along the $z-$axis in cylindrical coordinates $(r, \theta, z)$. Show that at large $r$ the amplitude can be approximated by $R(r) \sim 1 - p/r^2 + O(r^{-3})$ and specify the constant $p$ in terms of $\beta$ and $\mathcal{N}$ only.

**2** Consider the non-dimensional Gross-Pitaevskii equation

$$-2i\psi_t = \nabla^2\psi + (1 - |\psi|^2)\psi.$$

(i) Write down the linearised equations for the disturbances of the real and imaginary parts of $\psi = 1 + u + iv$ with respect to the ground state and find the differential equation for $u$ that does not depend on $v$. Consider the disturbances of the form $u = \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ and find the dispersion relationship $\omega(k)$.

(ii) Write down the equation for solitary waves moving with velocity $U$ in the positive $z$-direction in the frame of reference in which the solitary wave is stationary.

(iii) Given the energy

$$E = \frac{1}{2}\int |\nabla\psi|^2 \, dV + \frac{1}{4}\int (1 - |\psi|^2)^2 \, dV$$

and momentum

$$p = \frac{1}{2i}\int [(\psi^* - 1)\partial_z\psi - (\psi - 1)\partial_z\psi^*] \, dV,$$

show that

(a) $U = \partial E/\partial p$ where the the derivatives are taken along the sequence of solitary waves;

(b) $E = \int |\partial_z\psi|^2 \, dV$.

**3** Consider a *two-dimensional* Bose-Einstein condensate in a trap described by the Gross-Pitaevskii equation in polar coordinates $(r, \theta)$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{1}{2} m \omega^2 r^2 \psi + V_0 |\psi|^2 \psi,$$

where $\hbar$ is the Plank constant, $m$ is the particle mass, $\omega$ is the trap frequency, and $V_0 = 4\pi\hbar^2 a/m$ is the effective pair interaction, $a$ being the scattering length. The total number of particles in the trap is $N = \int |\psi(\mathbf{x})|^2 \, d\mathbf{x}$.

(i) Give a definition of the Thomas-Fermi (TF) regime in terms of the relationship between $N, a, \omega$. Find the approximation for the ground state of a condensate in the TF regime.

(ii) Calculate the energy of a vortex with winding number $\mathcal{N} = 1$ in the centre of the condensate in the TF regime. You can use the fact that the vortex energy in a uniform condensate is given by $E_v = \pi n_0 \frac{\hbar^2}{m} \mathcal{N}^2 [\log(L/\xi) + L_{0\mathcal{N}}]$, where $n_0$ is the density of the ground state, $\xi$ is the healing length, $L$ is the container radius, and $L_{0\mathcal{N}}$ are known constants.

[*Hint: you may assume that the condensate radius* $R \gg L \gg \xi$ .]

(iii) Find the total angular momentum of condensate with a vortex of winding number $\mathcal{N} = 1$ in the centre of the condensate.

(iv) Redo (ii) and (iii) for vortices of arbitrary $\mathcal{N}$.

**4** The effective Gross-Pitaevskii equation that describes the evolution of the exciton-polariton condensate with a wavefunction $\psi$ has the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_0|\psi|^2\psi + i(\gamma - \Gamma|\psi|^2)\psi,$$

where $\gamma$ and $\Gamma$ are positive constants representing the effective rate of creating new condensate particles and the particles' decay rate due to many particle collisions respectively, and $V_0$ is the effective pair interaction.

For the condensate in *equilibrium*:

(i) Write down the equation for the wavefunction.

(ii) Write down the hydrodynamical equations for the number density $n = |\psi|^2$ and the velocity potential $\phi$ and express $\nabla \cdot (n\nabla\phi)$ in terms of $\gamma, \Gamma$, $n$ and $\hbar$.

(iii) Find the wave function of the condensate assuming the condensate has constant number density and a constant velocity in the $x$ direction. Discuss the range of parameters for which such a solution exists.

(iv) The condensate in equilibrium is contained in an infinitely long right cylinder. The number of particles per unit length along the cylinder axis is $N$. Using the continuity equation, or otherwise, show that

$$\int_S n^2\, d\mathbf{x} = qN,$$

where $S$ is the cylinder cross-section orthogonal to the cylinder axis. You need to specify $q$ in terms of the parameters of the system.

[*Hint: note that* $n = 0$ *on the surface of the cylinder.*]

## END OF PAPER