UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 1.30 to 3.30

PAPER 83

ENVIRONMENTAL FLUID DYNAMICS

*You may attempt **ALL** questions, although high marks can be achieved by good answers to **THREE** questions.*

Completed answers are preferred to fragments.

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a steady hydraulic flow of water of depth $h(x)$ beneath air along a channel with rectangular cross-section of width $b(x)$ and bottom elevation $H(x)$.

(a) Starting from the momentum equation for an inviscid flow and conservation of mass, derive the specific energy function E for this flow and show how this is related to the Froude number F. Sketch the form of E and comment on the propagation of long waves. State any assumptions made.

(b) For a channel with $H = 0$ and $b = b_0\,(1 + (x/L)^2)$, determine the condition or conditions required for the flow to be hydraulically controlled, demonstrating that the control must be located at $x = 0$.

(c) Suppose water seeps out through the bottom of the channel at a rate S per unit length, independent of the channel width or water depth. Derive a modified specific energy function, assuming a volume flux Q_0 at $x = 0$, and determine the location of the hydraulic control, should one exist. Determine also the depth at the control.

(d) Describe how things would differ if the seepage S were negative so that water is added to the channel rather than removed. You need not repeat the analysis performed in part (c).

2 Consider a channel of uniform width and depth with triangular cross-section containing a layer of fluid of density ρ_1 beneath a deep ambient fluid of depth ρ_2.

(a) The flow is both Boussinesq and shallow water: what conditions must be satisfied for this to be the case? Determine the shallow water equations for this system and show that the system is hyperbolic with characteristics given by $\lambda = u \pm (g'h/2)^{1/2}$, where u is the velocity, h is the height of the interface above the lowest point in the cross-section, and g' is the reduced gravity. Determine the quantity or quantities conserved along the characteristics.

(b) For $t < 0$ the fluid of density ρ_1 is confined behind a dam located at $x = 0$ such that it is at rest with a depth h_0. Suppose the dam is removed at $t = 0$. Why is a constant Froude number condition of the form $u_f = F(g'h_f)^{1/2}$ necessary and appropriate for the front? Here, u_f is the speed of the front at which $h = h_f$. Outline, with the aid of a sketch, how an appropriate Froude number F could be determined: you do not need to determine an actual value for F.

(c) Determine the velocity field $u(x,t)$ and depth profile $h(x,t)$ of the current, assuming there is no drag and the channel is of infinite length. Express your answer in terms of $c_0 = (g'h_0/2)^{1/2}$.

(d) Describe, with the aid of sketches, how the evolution of the current is modified if the channel extends only as far as $x = -x_0$. Derive and solve a box model for the late-time evolution in this case.

3 Consider a steady turbulent flow of a homogeneous fluid above a rigid horizontal boundary where the mean velocity field is purely horizontal with a profile given by $\overline{\mathbf{u}} = (U(z), 0, 0)$ and the mean pressure field $\bar{p} = P(z)$ is hydrostatic.

(a) By averaging the momentum equation for a turbulent flow with velocity $\mathbf{u}(x,t) = (U + u', v', w')$, derive an ordinary differential equation for $U(z)$. What do the terms of the form $\overline{(\mathbf{u}' \cdot \nabla)u'}$ represent? State why we may neglect derivatives in the x and y directions in these terms, and show that the mean shear stress acting on the fluid is

$$\tau \;=\; \mu \frac{dU}{dz} - \rho \frac{d}{dz}\,\overline{u'w'}\,.$$

Over what length scale δ near the ground does viscosity dominate?

(b) By parameterising the turbulence using a mixing length model we may rewrite the equation for $U(z)$ in terms of an eddy viscosity $\nu_T = \alpha \hat{u} \ell$ for some constant α as

$$\tau \;=\; \nu_T \frac{dU}{dz}\,.$$

Take the turbulent velocity scale $\hat{u}$ as a constant and choose an appropriate scaling for ℓ. Determine the form of $U(z)$ for $z \gg \delta$ if there are no internal sources of momentum.

[Hint: consider the largest eddy that can exist below a height z.]

(c) Suppose the flow contains a dilute suspension of particles with volume concentration $\phi(z,t)$ and settling velocity $-W_s$. The turbulence is not sufficiently strong for the particles to be 'well mixed', but does drive a vertical turbulent flux given by

$$F_\phi \;=\; -\kappa_T \frac{\partial \phi}{\partial z}\,,$$

where $\kappa_T = \frac{1}{2}\nu_T$. What is the turbulent Schmidt number? Derive an equation for the evolution of $\phi(z,t)$.

(d) Describe the process by which sedimentation will occur, and give a criterion for resuspension. For the case when sedimentation and resuspension are in equilibrium, determine the equilibrium particle concentration $\phi(z)$ for $z \gg \delta$ given $\phi(1) = \phi_1$, and $\phi \to 0$ as $z \to \infty$.

4 The conservation of mass and momentum equations describing an axisymmetric Boussinesq, time-dependent, turbulent, buoyant plume with a top-hat profile in a stratified environment of density $\rho_0(z)$ may be written in terms of the plume velocity w, width b and density ρ as

$$\frac{\partial}{\partial t}(\rho b^2) + \frac{\partial}{\partial z}(\rho b^2 w) = 2\,\rho_0\, b u_e\,,$$

$$\frac{\partial}{\partial t}(\rho b^2 w) + \frac{\partial}{\partial z}(\rho b^2 w^2) = (\rho_0 - \rho)\, g b^2\,.$$

(a) State Batchelor's law for the entrainment velocity u_e. Derive an equation for volume conservation and hence show that conservation of buoyancy may be written in the form

$$\frac{\partial}{\partial t}((\rho_0 - \rho)\, g b^2) + \frac{\partial}{\partial z}((\rho_0 - \rho)\, g b^2 w) = \frac{d\rho_0}{dz}\, g b^2 w\,.$$

(b) Define the mass flux πQ, buoyancy flux πF and momentum flux πM, and rewrite the plume equations in terms of these. Determine whether or not the plume equations are a hyperbolic system.

(c) Show that a power law solution to the steady plume equations exists in a statically unstable stratification described by a constant buoyancy frequency with $N^2 = -G^2 < 0$, and determine this solution. Show that in this case the plume radius is given by $b = 2\alpha z/3$. [You may assume the static instability does not drive any flow in the ambient fluid.]

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