UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2007 9.00 to 11.00

PAPER 80

CLASSICAL WAVE SCATTERING

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a 3-dimensional wave guide which has an exit aperture A in the (y, z) plane at $x = 0$. An electromagnetic wave travelling in the positive x-direction emerges from the waveguide through a screen at $x = 0$, which polarizes the wave so that the x and z components vanish.

Find an expression for the field in $x > 0$ beyond the end of the waveguide as a superposition of plane waves. You may use the fact that the field beyond the aperture of the waveguide can be written as

$$\mathbf{E}(x) = \nabla \times \int_A (\mathbf{n} \times \mathbf{E}_{inc}(x, y, z))\ G\ dydz \quad ,$$

where G is the scalar free space Green's function.

(b) Consider now a generic electromagnetic wave, emerging from the waveguide as before, but without being polarised. Show how this wave can also be expressed as a superposition of plane waves.

2 Given a 2-dimensional space with horizontal coordinate x and vertical coordinate z, let $z \geqslant f(x)$, where $f(x)$ is an irregular boundary, define a half-space occupied by a medium with density ρ and sound speed c. The half-space $z < f(x)$ is a vacuum. Any time-harmonic scalar wave potential $\psi e^{-i\omega t}$ in the medium obeys the Helmholtz wave equation $(\nabla^2 + k^2)\psi = 0$.

A source in the half-space $z \geqslant f(x)$ gives rise to a known time-harmonic incident field ψ_{inc}, which impinges on the boundary, causing a scattered field ψ_{sc} such that $\psi = \psi_{inc} + \psi_{sc}$.

(a) Denote by $\hat{\psi}_{sc}(\nu)$ the Fourier Transform of ψ_{sc} with respect to x along the horizontal plane $z = 0$, and express ψ_{sc} at an arbitrary point (x, z) in terms of $\hat{\psi}_{sc}(\nu)$.

(b) Find the autocorrelation function of the scattered field

$$m(\xi, z) = < \psi_{sc}(x_1, z), \psi_{sc}(x_2, z) >, \qquad \text{where } \ \xi = (x_1 - x_2) \ ,$$

as a function of z, in terms of $\hat{\psi}(\nu)$.

(c) Suppose now that the surface is only slightly rough, i.e. it satisfies $kf(x) \ll 1$, and a plane wave is incident on it with Dirichlet boundary conditions.

Derive an approximation for the quantity $\hat{\psi}(\nu)$ using the first order perturbation solution.

Hence, show that the mean scattered field is specular.

3 (a) A time-harmonic scalar wave $\psi e^{-i\omega t}$ is a superposition of plane waves propagating with wavenumber k at small angles to the horizontal in a uniform 2-dimensional medium (x, z), where x is the horizontal and z the vertical coordinate.

Derive the parabolic wave equation for the slowly-varying part E of ψ.

(b) Suppose that a horizontally propagating plane wave encounters a thin vertical layer of thickness δ at $x = -\delta$, which imposes a random phase factor $e^{i\phi(z)}$ on the wave, where ϕ is normally distributed, statistically stationary with respect to z, and has variance σ^2.

(i) Find the first moment of the Fourier Transform of the field E, $< \hat{E}(x, \nu) >$, as a function of x, for $x \geqslant 0$.

(ii) Hence, show that the mean field $< E(x, z) >$ and the mean intensity $< I(x, z) >=< |E(x, z)|^2 >$ are constant with respect to distance x.

(c) Suppose now that the phase imposed by the layer is a deterministic function $\phi = \cos z$. Find an approximate expression to second order in x for the intensity $I(x, z) = |E(x, z)|^2$ at a small distance x from the screen.

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Let an inverse problem be expressed by the equation $Ax = y$, where A is a given linear operator $A : X \mapsto Y$, from a normed space X into a normed space Y, and y is known.

(a) Explain briefly what is meant by saying that this problem is ill-posed.

(b) Consider now the inverse scattering problem in 3 dimensions, in which a known incident field $u_i(\mathbf{x})$ is scattered by a scatterer D with Dirichlet boundary conditions on ∂D, giving rise to a far-field pattern whose measured value is given by $u_\infty^{(\delta)}(\mathbf{x})$. The unknown scattered field $u_s(\mathbf{x})$ satisfies the Helmholtz equation and the Sommerfeld conditions, and for large $|\mathbf{x}|$ can be expanded in terms of spherical harmonics:

$$u_s(\mathbf{x}) = k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^{n+1} a_n^m H_n^{(1)}(k\|\mathbf{x}|) Y_n^m \left(\tfrac{\mathbf{x}}{|\mathbf{x}|}\right) \ ,$$

where $H_n^{(1)}$ is a spherical Hankel function of the first kind of order n. The error in the measured far-field pattern is such that $\| u_\infty^{(\delta)}(\mathbf{x}) - u_\infty(\mathbf{x}) \| \leqslant \delta$.

(i) Show that the far-field pattern $u_\infty(\mathbf{x})$ of the scattered field $u_s(\mathbf{x})$ is given by

$$u_\infty(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m \ ,$$

with the coefficients a_n^m satisfying

$$\sum_{n=0}^{\infty} \left[\left(\tfrac{2n}{ek|\mathbf{x}|}\right)^{2n} \sum_{m=-n}^{n} |a_n^m|^2 \right] \leqslant \infty \qquad \text{for large } |\mathbf{x}| \ . \tag{1}$$

(ii) The measured far-field pattern $u_\infty^{(\delta)}$ can also be written as

$$u_\infty^{(\delta)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m Y_n^m \ .$$

Explain how small errors in the measured far field pattern can be amplified and result in arbitrarily large errors in the estimate of the scatterer.

In answering the questions above, you may use the asymptotic expression of the Hankel function

$$H_n^{(1)}(y) = O\left(\tfrac{2n}{ey}\right)^n \qquad \text{as } n \to \infty \ ,$$

and Parseval's equality for the coefficients of the expansions over an orthonormal basis. This states that, if $\mathbf{v}_n$ is an orthonormal basis in an Hilbert space H, then the identity $\| \mathbf{y} \|^2 = \sum_n |(\mathbf{y}, \mathbf{v}_n)|^2$ is satisfied $\forall \mathbf{y} \in H$.

(c) Illustrate how Tikhonov regularisation can be applied in the case considered in (b), and use it to put a bound on the error in the solution.

How should α be chosen in order to minimise the effect of the error in the measured far-field pattern? Will minimising this effect achieve the optimal solution?

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