UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 9.00 to 12.00

PAPER 8

INTRODUCTION TO FUNCTIONAL ANALYSIS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the Baire Category theorem and use it to prove the principle of uniform boundedness.

Let $\mathcal{A}$ be the array a_{ij} with $a_{ij} \in \mathbb{R}$ for $i, j \geqslant 1$. We say that a sequence $\mathbf{x} = (x_1, x_2, \ldots)$ with $x_j \in \mathbb{R}$ has $\mathcal{A}$ limit $\mathcal{A}(\mathbf{x})$ if the sum

$$\mathcal{A}_i(\mathbf{x}) = \sum_{j=1}^{\infty} a_{ij} x_j$$

exists and $\mathcal{A}_i(\mathbf{x}) \to \mathcal{A}(\mathbf{x})$ as $i \to \infty$. We say that $\mathcal{A}$ is regular if, whenever $x_j \to x_0$ as $j \to \infty$, the sequence $\mathbf{x}$ has $\mathcal{A}$ limit x_0. Show that $\mathcal{A}$ is regular if and only if the following three conditions hold.

(α) There exists an M such that $\sum_{j=1}^{\infty} |a_{ij}| \leqslant M$ for all i.

(β) $a_{ij} \to 0$ as $i \to \infty$ for each j.

(γ) $\sum_{j=1}^{\infty} a_{ij} \to 1$ as $i \to \infty$.

Verify that if $\mathcal{B} = b_{rs}$ with $b_{rs} = r^{-1}$ for $1 \leqslant s \leqslant r$ and $b_{rs} = 0$ otherwise, then conditions (α), (β) and (γ) hold so $\mathcal{B}$ is regular. Find a bounded sequence $\mathbf{w}$ which has no $\mathcal{B}$ limit and prove it has this property.

By using conditions (α), (β) and (γ), or otherwise show that, if $\mathcal{A}$ is regular, then we can find a bounded sequence $\mathbf{y}$ such that $\mathbf{y}$ has no $\mathcal{A}$ limit.

Consider the space l^∞ of bounded sequences with the standard norm $\|\mathbf{x}\| = \sup_n |x_n|$. Show that, if $\mathcal{A}$ is regular, then

$$E_{\mathcal{A}} = \{\mathbf{x} \in l^\infty \,:\, \text{x has a } \mathcal{A} \text{ limit}\}$$

is closed and nowhere dense in l^∞. Deduce that, given a countable collection $\mathcal{A}^{[m]}$ $[m = 1, 2, \ldots]$ we can find a bounded sequence $\mathbf{u}$ such that $\mathbf{u}$ has no $\mathcal{A}^{[m]}$ limit for any $m \geqslant 1$.

By using the Bolzano-Weierstrass theorem, or otherwise, show that given any bounded sequence $\mathbf{x}$ we can find a regular $\mathcal{A}$ such that $\mathbf{x}$ has a $\mathcal{A}$ limit.

2 (i) Let V be a real normed space with a countable dense subset $\mathbf{y}_1, \mathbf{y}_2, \ldots$. Show *without using the the axiom of choice* that if U is a subspace of V and $T : U \to \mathbb{R}$ is a continuous linear function, we can find a continuous linear function $\tilde{T} : V \to \mathbb{R}$ such that $\tilde{T}(\mathbf{u}) = T(\mathbf{u})$ for all $\mathbf{u} \in U$ and $\|\tilde{T}\| = \|T\|$.

(ii) Define the extreme points of a set in a real vector space. Give an example of a non-empty convex set in $\mathbb{R}^2$ with no extreme points. Let l^∞ be the space of bounded real sequences, l^1 the space of absolutely convergent real sequences and l^2 the space of square summable real sequences, each with their usual norms. Find the extreme points of the closed unit ball for each case proving your statements. (You may find it useful to think about the two dimensional analogues.)

[The two parts of the question are not related. The first part carries twice the weight of the second.]

3 Prove the theorem of Gelfand-Mazur. (If you use properties of the resolvent or spectrum you should prove them.) Develop the theory of maximal ideals to the point where you can establish a bijection with multiplicative linear functionals.

4 Suppose that B is a commutative Banach algebra with unit with maximal ideal space $\mathcal{M}$. Show that, if B has an involution $f \mapsto f^*$, then the Gelfand transform $B \to C(\mathcal{M})$ is a norm preserving isomorphism with $(f^*)\hat{} = (f\hat{})^-$.

Let V be a finite dimensional complex inner product space. Use the result of the first paragraph to show that, if $\alpha, \beta : V \to V$ are linear maps and α, β and the adjoints α^*, β^*, all commute then we can find a set $\pi_1, \pi_2, \ldots \pi_m$ of commuting projections together with λ_j, $\mu_j \in \mathbb{C}$ such that

$$\alpha = \sum_{j=1}^{m} \lambda_j \pi_j \text{ and } \beta = \sum_{j=1}^{m} \mu_j \pi_j .$$

END OF PAPER