UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 9.00 to 12.00

PAPER 79

PERTURBATION AND STABILITY METHODS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Find for $\epsilon \to 0$ the leading order term of an asymptotic expansion for each root of the equation

$$x^3 - \epsilon x^2 + 2\epsilon^3 x + 2\epsilon^6 = 0.$$

Find also the first correction for the root of smallest magnitude.

(b) Find the first three terms of an asymptotic expansion for

$$f(x) = \int_0^x t^{-1/2} e^{-t} dt$$

when $x \to \infty$ with x real.

(c) Consider the function

$$g(x) = \int_0^1 \log t \, e^{ixt} \, dt$$

in the limit $x \to \infty$ with x real. By using the steepest descent contour, or otherwise, find the full asymptotic expansion for $g(x)$.

[*Watson's lemma may be quoted without proof.*
$\int_0^\infty e^{-u} \log \; u \, du = -\gamma$ *where* γ *is Euler's constant.*]

2 Find the exact solution of the equation (for $t > 0$)

$$\ddot{x} + 2\epsilon\dot{x} + x = 0 \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1.$$

If $\epsilon \to 0$, obtain from your solution:

(a) the Poincaré expansion $y(\epsilon, t)$ of $x(t)$ with errors of order ϵ^2;

(b) an asymptotic expansion $z(\epsilon, t)$ of $x(t)$ with errors of order ϵ^2 that remains valid when $\epsilon t = ord(1)$, but not when $\epsilon^2 t = ord(1)$.

Sketch graphs that illustrate the differences between $y(\epsilon, t)$ and $z(\epsilon, t)$ and between $z(\epsilon, t)$ and $x(t)$ for fixed $\epsilon > 0$.

The displacement $x(t)$ of a pendulum that suffers weak air resistance satisfies the equation for $t > 0$

$$\ddot{x} + \epsilon|\dot{x}|\dot{x} + x = 0 \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1.$$

For $\epsilon \to 0$ use the method of multiple scales to find a leading order approximation for $x(t)$ valid for $\epsilon t = ord(1)$.

Give sketches of your solution for both $\epsilon > 0$ and the unphysical case $\epsilon < 0$. Comment on the range of validity in t and the size of the error term for both $\epsilon > 0$ and $\epsilon < 0$.

3 The function $y(x)$ satisfies the equation

$$\varepsilon \frac{d^2y}{dx^2} + \left(1 + \frac{2\varepsilon}{x} - \frac{2\varepsilon^3}{x^2}\right)\frac{dy}{dx} + \frac{2y}{x} = 0\,,$$

where $\varepsilon > 0$, together with the boundary conditions

$$y(0) = \gamma \quad \text{and} \quad y(1) = \varepsilon^3\,,$$

where γ is a constant.

If $\varepsilon \ll 1$ find the order one value of γ for which an asymptotic solution can be found such that $y(x)$ is no larger than order one for $0 \leqslant x \leqslant 1$.

Briefly comment on whether the problem as posed specifies a unique solution.

Hints. Note that

$$y_{xx} + \left(1 + \frac{2}{x}\right) y_x + \frac{2y}{x} = \frac{1}{x^2}\left(x^2 y_x + x^2 y\right)_x\,,$$

and that the general solution to

$$y_{xx} + \left(\frac{2}{x} - \frac{2}{x^2}\right) y_x = 0\,,$$

is

$$y = A\,exp\left(-\frac{2}{x}\right) + B\,,$$

where A and B are constants.

4 (a) Consider inviscid fluid flow between rigid walls at $y = -1$ and $y = 1$. Initially the velocity profile is given by $(U(y), 0, 0)$. Suppose now that the flow is perturbed so that

$$\mathbf{u} = (U(y), 0, 0) + (u, v, w)\,.$$

Derive linearised governing equations for the velocity v and the vorticity $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$.

Suppose that the perturbations have a single Fourier component so that

$$(v, \eta) = (\widetilde{v}(y,t), \widetilde{\eta}(y,t))exp(i(\alpha x + \beta z))\,,$$

and let $\widetilde{v}_0(y) = \widetilde{v}(y, 0)$ and $\widetilde{\eta}_0(y) = \widetilde{\eta}(y, 0)$. If $U = \lambda y$ and $\alpha \neq 0$, then by means of a Laplace transform, or otherwise, find an integral expression for $\widetilde{v}$. Also find an an integral expression for $\widetilde{\eta}$ in terms of $\widetilde{v}$.

Consider separately the case when $\alpha = 0$ and $\beta \neq 0$. Solve for $\widetilde{\eta}$, and comment on your result.

(b) Consider instead the inviscid flow, $(U(y), 0, 0)$, of a stratified fluid with density $\rho(y)$. Again assume that there are rigid walls at $y = -1$ and $y = 1$. In the so-called *Boussinesq limit*, it may be shown that the equation governing linear two-dimensional perturbations to this flow profile and density profile is

$$(U - c)\left(D^2 - \alpha^2\right)\phi - U''\phi + \frac{J(y)\phi}{U - c} = 0\,,$$

where $D = \frac{d}{dy}$, $U'' = \frac{d^2U}{dy^2}$,

$$\mathbf{u} = (U(y), 0, 0) + \left(\frac{d\phi}{dy}, -i\alpha\phi, 0\right) exp(i\alpha(x - ct)) + \ldots\,,$$

and

$$J(y) = -\frac{1}{\rho}\frac{d\rho}{dy}\,.$$

If $H = (U - c)^{-\frac{1}{2}}\phi$, show that

$$D((U - c)DH) - \left(\alpha^2(U - c) + \tfrac{1}{2}U'' + \frac{\frac{1}{4}U'^2 - J}{U - c}\right) H = 0\,.$$

Hence deduce that if the flow is unstable, then somewhere in the flow

$$J < \tfrac{1}{4}U'^2\,.$$

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