UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 9.00 to 12.00

PAPER 77

FUNDAMENTALS OF ATMOSPHERE–OCEAN DYNAMICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail.

(x, y, z) denotes right-handed Cartesian coordinates and (u, v, w) the corresponding velocity components; t is time; the gravitational acceleration is $(0, 0, -g)$ where g is a positive constant; $\hat{z} = (0, 0, 1)$ is a unit vector directed vertically upward.
The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N denotes the buoyancy frequency of a stratified fluid.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State what is meant by the Boussinesq approximation for a continuously stratified ideal fluid, and what is meant by the 'buoyancy acceleration'. Define the buoyancy frequency $N(z)$. Write down the nonlinear Boussinesq momentum, mass-continuity and buoyancy equations for a non-rotating reference frame, carefully explaining the meanings of the symbols used.

An ideal Boussinesq fluid with constant N undergoes small-amplitude motion in the region $x > 0$, in response to the motion of a boundary whose undisturbed position is the vertical plane $x = 0$. The velocity of the boundary is prescribed as (with real part understood)

$$u(0, z, t) = \epsilon \exp\left(imz - i\omega t\right) ,$$

where ϵ, m and ω are real positive constants, ϵ being considered small. Starting from the equations linearized about rest, derive the dispersion relation for plane-wave disturbances and use it to find the response to the boundary motion when (i) $\omega < N$, and (ii) $\omega > N$. In each case, give expressions for the velocity-field components (u, w) and for the buoyancy-acceleration anomaly field σ.

Suppose now that the velocity of the boundary is prescribed as (again with real part understood)

$$u(0, z, t) \;=\; \epsilon\, u_0(z)\, e^{-i\omega t} \quad \text{where} \quad u_0(z) \;=\; \int_0^\infty \left\{\hat{u}_0(m) e^{imz} + \hat{u}_0^*(m) e^{-imz}\right\} dm$$

and where $0 < \omega < N$. Suppose that $\hat{u}_0(m)$ and its complex conjugate $\hat{u}_0^*(m)$ are chosen such that the real-valued function $u_0(z)$ is negligible outside some neighbourhood of $z = 0$, i.e., negligible for $|z| > H$, say. Show that the response has horizontal velocity component

$$u(x,\, z,\, t) \;=\; f(Z_1)\cos(\omega t) + g(Z_1)\sin(\omega t) + f(Z_2)\cos(\omega t) - g(Z_2)\sin(\omega t)$$

where $Z_1 = z + \alpha x$, $Z_2 = z - \alpha x$, and where

$$\alpha \;=\; \left(\frac{\omega^2}{N^2 - \omega^2}\right)^{1/2} \;>\; 0 \;,$$

$$f(z) \;=\; \epsilon \,\mathrm{Re} \int_0^\infty \hat{u}_0(m)\, e^{imz}\, dm \;,$$

$$g(z) \;=\; \epsilon \,\mathrm{Im} \int_0^\infty \hat{u}_0(m)\, e^{imz}\, dm \;.$$

Assuming some simple shape for $u_0(z)$ in $|z| < H$, sketch the velocity field at the instant when the boundary is instantaneously flat and moving fastest. Briefly state how that velocity field differs qualitatively from the velocity field at other instants.

If instead the boundary motion is $u(0, z, t) = \epsilon\, u_0(z)\, T(t)$, where $T(t)$ is a pulse (zero outside some small time interval), **briefly** describe the qualitative appearance of the response. [*Do not solve this problem. Argue qualitatively from your knowledge of the dispersion properties.*]

2 Write down the fully nonlinear Boussinesq momentum equation for two-dimensional motion ($\partial/\partial y = 0$) of an ideal stably stratified, non-rotating fluid. Using the mass-continuity equation, derive the y-component of the vorticity equation in the form

$$\nabla^2\psi_t + \sigma_x = J(\psi, \nabla^2\psi)$$

where $\psi(x, z, t)$ is a streamfunction to be specified. Subscripts denote partial differentiation, σ is the departure of the buoyancy acceleration from its background value, and J is the Jacobian with respect to x and z. Write down the corresponding nonlinear equation for σ, again using the streamfunction ψ.

Linearize the equations about a background state in which the buoyancy frequency N is a function of z, and in which there is a steady flow in the positive x direction with velocity $\bar{u}(z)$. Show that disturbances of the form $\psi' = \hat{\psi}(z)\exp\bigl(ik(x - ct)\bigr)$ satisfy the Taylor–Goldstein equation

$$\hat{\psi}_{zz} + \left(\ell^2 - k^2\right)\hat{\psi} = 0 \qquad \text{where} \qquad \ell^2(z) = \frac{N^2(z)}{(\bar{u} - c)^2} - \frac{\bar{u}_{zz}}{(\bar{u} - c)}\ .$$

What restoring mechanisms are represented by the two terms in the definition of $\ell^2(z)$?

If the flow is confined between rigid horizontal boundaries $z = 0$, $z = \pi H$, show that

$$k^2 = I(\hat{\psi}; c) \qquad \text{where} \qquad I(\hat{\psi}; c) = \frac{\int_0^{\pi H}(\ell^2\hat{\psi}^2 - \hat{\psi}_z^2)\,dz}{\int_0^{\pi H}\hat{\psi}^2 dz}\ .$$

By using the stationarity property of $I(\hat{\psi}; c)$ with respect to $\hat{\psi}$ [*do not prove this*] deduce that the group velocity c_{g} in the x direction is given by

$$c_{\mathrm{g}} = c + \frac{2k^2\int_0^{\pi H}\hat{\psi}^2 dz}{\int_0^{\pi H}\{(\bar{u} - c)^{-3}N^2 + (\bar{u} - c)^{-1}\ell^2\}\hat{\psi}^2 dz}\ .$$

If $N(z) = N_0\sin(z/H)$ and $\bar{u}(z) = U_0\sin(z/H)$, where N_0 and U_0 are positive constants, show that with suitably chosen c the Taylor–Goldstein equation has solutions of the form $\hat{\psi}(z) \propto \sin(nz/H)$ where $n = 1, 2, 3, \ldots$. Show that k is always real for at least one of these values of n. Deduce the values of k^2 and c_{g} in that case.

Show that when N and $\bar{u}$ are constant the same functions $\hat{\psi}(z)$ can still be solutions, with the same c, though with different values of k^2 to be determined. Is there still a value of n for which k is always real?

Determine which, if any, of your solutions are also exact solutions of the fully nonlinear equations.

3 Derive the shallow-water equations in a **non-rotating** fluid system with a flat bottom boundary and layer depth $h(x,\, y,\, t)$, carefully explaining the assumptions made.

State how the equations should be modified to allow for rotation of the frame of reference, with constant angular velocity $(0, 0, \frac{1}{2}f)$. Derive the vertical component of the vorticity equation from the momentum equations, and show that Rossby's potential vorticity q_a/h is an exact material invariant, i.e. that

$$\frac{\mathcal{D}_H}{\mathcal{D}t}\left(\frac{q_a}{h}\right) \;=\; 0\;, \tag{$*$}$$

where $q_a = f + \partial v/\partial x - \partial u/\partial y$ and where $\mathcal{D}_H/\mathcal{D}t = \partial/\partial t + \mathbf{u}_H \cdot \nabla_H$, with $\mathbf{u}_H = (u, v, 0)$ and $\nabla_H = (\partial/\partial x,\, \partial/\partial y,\, 0)$.

Using order-of-magnitude arguments and making appropriate assumptions about small parameters, carefully derive the quasi-geostrophic counterpart to $(*)$. Express everything in terms of an appropriate streamfunction $\psi(x, y, t)$, to be specified, and an appropriate length scale L_R. Briefly explain what is meant by (a) 'potential-vorticity conservation' and (b) 'potential-vorticity inversion', in this quasi-geostrophic context.

A unidirectional current $\bar{u} = -\partial\bar{\psi}(y)/\partial y$ flows in the positive x direction, and the corresponding quasi-geostrophic potential vorticity is $\bar{Q}(y)$. Show that a small disturbance ψ', Q' satisfies

$$Q'_t \;+\; \bar{u}(y)Q'_x \;+\; \bar{Q}_y\psi'_x \;=\; 0\;, \tag{$\dagger$}$$

where subscripts denote partial differentiation.

Now suppose that $\bar{u}(y) \to 0$ as $|y| \to \infty$, and that $\bar{Q}$ has one constant value for all $y > 0$ and another, smaller constant value for all $y < 0$, with a jump discontinuity $\big[\bar{Q}\big]^+_-$ at $y = 0$. Show that $\bar{u}(y) \propto \exp(-|y|/L_R)$ and find the constant of proportionality, $\bar{u}(0)$, in terms of $\big[\bar{Q}\big]^+_-$. If the small disturbance takes the form $\psi' = \hat{\psi}(y)\exp\big(ik(x - ct)\big)$, with k and c constant, show that it satisfies $(\dagger)$ if

$$\hat{\psi}_{yy} \;-\; k^2\hat{\psi} \;-\; L_R^{-2}\hat{\psi} \;=\; 0 \qquad \text{for} \qquad |y| > 0$$

and

$$\big(\bar{u} - c\big)\,\big[\hat{\psi}_y\big]^+_- \;+\; \big[\bar{Q}\big]^+_-\,\hat{\psi} \;=\; 0 \qquad \text{at} \qquad y = 0\;.$$

Find $\hat{\psi}(y)$ and deduce that

$$c \;=\; \bar{u}(0)\left\{1 - \left(\frac{L_R^{-2}}{k^2 + L_R^{-2}}\right)^{1/2}\right\}\;.$$

What is the wave propagation mechanism involved? On what property of $\bar{Q}(y)$ does it depend?

4 For a Boussinesq, stably stratified ideal fluid on a rotating Earth, derive the equation governing the quasi-geostrophic potential vorticity

$$Q \;=\; f + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) \;,$$

taking as your starting point the buoyancy and vorticity equations in the form

$$\frac{\mathcal{D}_{\mathrm{g}}}{\mathcal{D}t}\left(f_0\frac{\partial\psi}{\partial z}\right) + N^2 w = 0$$

and

$$\frac{\mathcal{D}_{\mathrm{g}}}{\mathcal{D}t}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] + \beta\,\frac{\partial\psi}{\partial x} \;=\; f_0\,\frac{\partial w}{\partial z} \;,$$

with $f = f_0 + \beta y$, in the notation of the lecture notes. Do **not** give the full order-of-magnitude analysis, but briefly explain the meaning of the symbols ψ and $\dfrac{\mathcal{D}_{\mathrm{g}}}{\mathcal{D}t}$, showing why the buoyancy-acceleration anomaly $\propto \partial\psi/\partial z$. Through what approximate balance of terms is ψ related to the velocity field? Briefly state the conditions on dimensionless parameters that tend to favour such balance, and briefly explain why only the vertical component of the Earth's rotation rate enters the dynamics. Show that for a nearly-horizontal lower boundary with small elevation $z = b(x, y)$ the boundary condition on ψ is

$$\frac{\mathcal{D}_{\mathrm{g}}}{\mathcal{D}t}\left(\frac{f_0}{N^2}\frac{\partial\psi}{\partial z} + b\right) = 0 \quad \text{at} \quad z = 0 \;.$$

An ideal Boussinesq stratified, rotating fluid with constant N and $\beta = 0$ (i.e., $f = f_0 =$ constant) is unbounded above, and bounded below by a rigid boundary whose shape is given by

$$z \;=\; b(x, y) \;=\; \epsilon\,\cos(\ell y)\,\exp(ikx) \quad \text{with the real part understood,}$$

where ϵ, k and ℓ are real constants and ϵ is small. A steady, unidirectional mean current

$$\bar{u} \;=\; U(z) \;=\; U_0 + \Lambda z \;,$$

with U_0 and Λ constant, flows over the boundary undulations. The resulting small-amplitude disturbance has quasi-geostrophic streamfunction ψ'. Derive the linearized boundary condition on ψ' at $z = 0$. Show that a steady disturbance $\psi' = \psi'(x, y, z)$ is possible when $KU_0 + \Lambda \neq 0$, where $K = N(k^2 + \ell^2)^{1/2}/f > 0$, and find an explicit expression for ψ'. In the case $KU_0 + \Lambda > 0$, show by a sketch where the minima and maxima of the y-component of velocity occur relative to the minima and maxima of the topography $b(x, 0)$ along the x-axis.

When $KU_0 + \Lambda = 0$ show that ψ' must be unsteady, $\psi' = \psi'(x, y, z, t)$, and find a solution that satisfies the initial condition $\psi'(x, y, z, 0) = 0$. Deduce that

$$\frac{\partial\psi'}{\partial t} \;=\; \frac{ik\epsilon N^2 U_0}{Kf}\,\cos(\ell y)\,\exp(-Kz)\,\exp(ikx)$$

with the real part understood. Show by another sketch the relative positions, along the x-axis, of the minima and maxima of the y-component of velocity when $U_0 = -\Lambda/K > 0$. Briefly state a physical reason why the disturbance amplitude grows without bound.

END OF PAPER