UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 1.30 to 4.30

## PAPER 76

## NONLINEAR CONTINUUM MECHANICS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Isotropic elastic material has constitutive relation $\boldsymbol{\sigma} = \boldsymbol{\phi}(\mathbf{F})$, where $\boldsymbol{\sigma}$ is Cauchy stress and $\mathbf{F}$ is the deformation gradient. Prove that $\boldsymbol{\sigma}$ is coaxial with $\mathbf{FF}^T$ (or equivalently, with the left stretch matrix).

If the material is subjected to the simple shear deformation

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

(the irrelevant 3-components being disregarded), show that

$$\sigma_{11} - \sigma_{22} = \gamma\sigma_{12},$$

regardless of the detailed form of $\boldsymbol{\phi}$.

[*Express* $\sigma_{ij}$ *in terms of the principal stresses. There is no need to calculate the principal stretches; you may or may not find it convenient to do so, in calculating the principal axes of* $\mathbf{FF}^T$*.*]

**2** Incompressible material is reinforced by fibres which are aligned with the $X_1$-axis in the undeformed configuration, that render the material inextensible in the direction of the fibres. The material is deformed under the condition of plane strain, so that $\mathbf{X} \to \mathbf{x}$, where

$$\begin{aligned} x_1 &= x_1(X_1, X_2), \\ x_2 &= x_2(X_1, X_2), \\ x_3 &= X_3. \end{aligned}$$

Deduce that

$$x_{1,1} = \cos\theta, \quad x_{2,1} = \sin\theta,$$

where $\theta$ may depend on $X_1$ and $X_2$ and defines the direction of the fibres in the deformed configuration. [*The notation* $\phi_{,j} = \frac{\partial \phi}{\partial X_j}$ *for any function* $\phi(\mathbf{X})$ *is employed.*] Show that

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix},$$

where $\gamma = x_{1,2}\cos\theta + x_{2,2}\sin\theta$.

Deduce that

$$\gamma_{,1} = \theta_{,1} \text{ and hence that } \gamma = \theta + f(X_2)$$

for some function $f$. By considering $d\theta/ds$ along a curve $(X_1(s), X_2(s))$, show that

$$\theta = \text{constant} \;\; \text{along any curve for which } \frac{dX_2}{dX_1} = -\frac{1}{\gamma}.$$

Explain why any such deformation is possible with zero body force in such material, and how complete determination of the stress requires only the constitutive specification of the shear stress $\tau$ as a function(al) of $\gamma$ under the condition of simple shear.

**3** A cylinder composed of homogeneous isotropic elastic incompressible material occupies the domain $a_0^2 < X_1^2 + X_2^2 < b_0^2,\ 0 < X_3 < h_0$ in its unstressed reference configuration. It has energy function per unit reference volume $W(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i,\ i = 1, 2, 3$ denote the principal stretches. It is subjected to simultaneous twist and inflation, which can be viewed as first, the twist $\mathbf{X} \to \mathbf{y}$:

$$\begin{aligned} y_1 &= X_1 \cos(\alpha X_3) - X_2 \sin(\alpha X_3), \\ y_2 &= X_1 \sin(\alpha X_3) + X_2 \cos(\alpha X_3), \\ y_3 &= X_3, \end{aligned}$$

followed by the inflation $\mathbf{y} \to \mathbf{x}$:

$$\begin{aligned} x_1 &= f(\rho) y_1/\rho, \\ x_2 &= f(\rho) y_2/\rho, \\ x_3 &= y_3, \end{aligned}$$

where $\rho = (y_1^2 + y_2^2)^{1/2} \equiv (X_1^2 + X_2^2)^{1/2}$ and $f(\rho) = (\rho^2 + a^2 - a_0^2)^{1/2}$. Thus, the inner surface is inflated to radius $a$, and $\alpha$ is the angle of twist per unit height. The outer curved surface is traction-free.

Calculate the principal stretches $\lambda_i(\rho)$ at one representative location (such as $y_1 = \rho,\ y_2 = 0$), and deduce an expression for the total energy stored per unit height. The end couple has moment $M$ about the 3-axis and the internal pressure is $p(a)$. By considering the global balance of work-rate, show that

$$\begin{aligned} M &= 2\pi \frac{\partial}{\partial \alpha} \int_{a_0}^{b_0} \rho W(\lambda_1, \lambda_2, \lambda_3)\, d\rho, \\ p(a) &= \frac{1}{a} \frac{\partial}{\partial a} \int_{a_0}^{b_0} \rho W(\lambda_1, \lambda_2, \lambda_3)\, d\rho. \end{aligned}$$

Find $p(a)$ explicitly, for the case of neo-Hookean material for which $W(\lambda_1, \lambda_2, \lambda_2) = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$.

**4** (a) Write down the integral forms of the balance of energy and the entropy inequality, in the Lagrangian description, for a body with internal energy per unit mass $U(\mathbf{F},\eta,\xi)$, where $\mathbf{F}$ is the deformation gradient, $\eta$ is entropy per unit mass and $\xi$ represents a collection of internal variables $\{\xi_r\}$. Deduce (under the usual assumptions) the constitutive relations

$$P_{Ii} = \rho_0 \frac{\partial U}{\partial F_{iI}}, \quad \theta = \frac{\partial U}{\partial \eta},$$

where $\rho_0$ is the mass density in the undeformed configuration, $\mathbf{P}$ denotes the nominal stress tensor, and $\theta$ is the temperature. Deduce also that

$$\rho_0 \theta \dot{\eta} = \rho_0 r - q_{I,I} + f_r \dot{\xi}_r,$$

where $r$ is heat supply per unit mass, $\mathbf{q}$ is the nominal (or Lagrangian) heat flux vector, and $f_r = -\rho_o \partial U/\partial \xi_r$. Deduce also the inequality

$$f_r \dot{\xi}_r - \frac{q_I \theta_{,I}}{\theta} \geqslant 0.$$

(b) The specific free energy $\psi$ is defined so that $\psi(\mathbf{F},\theta,\xi) = U(\mathbf{F},\eta,\xi) - \theta\eta$. Consider the particular case

$$\psi = \psi(\mathbf{F}^*, \theta),$$

with $\mathbf{F}^* = \mathbf{F}\mathbf{A}^{-1}$: the internal variables $\xi_r$ are now replaced by $\{A_{JI}\}$, and $f_r$ are replaced by $Q_{IJ} = -\rho_o \partial\psi/\partial A_{JI}$. Find $\mathbf{P}$ and $\mathbf{Q}$ in terms of $\mathbf{P}^* = \rho_0 \partial\psi/\partial \mathbf{F}^*$.

For an isothermal process, given the dissipation potential $\Omega(\mathbf{Q},\mathbf{A})$, we have $\dot{A}_{JI} = \partial\Omega/\partial Q_{IJ}$. Find $A_{JI}$ as a function of time, in terms of the history of $\mathbf{Q}$, in the case that

$$\Omega(\mathbf{Q},\mathbf{A}) = \frac{\alpha}{n+1}\|\mathbf{Q}\|^{n+1} - \frac{1}{\tau} A_{JI} Q_{IJ}; \quad \|\mathbf{Q}\| = (Q_{IJ} Q_{IJ})^{1/2},$$

where $\alpha$, $\tau$ and $n$ are positive constants. Deduce a corresponding expression for the second Piola–Kirchhoff stress $\mathbf{T} = \mathbf{P}\mathbf{F}$, as a functional of $\mathbf{Q}$.

**5** Develop the formula

$$\frac{\delta\boldsymbol{\tau}}{\delta t} = \dot{\boldsymbol{\tau}} - \mathbf{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}^T$$

for the "upper-convected" rate of Kirchhoff stress $\boldsymbol{\tau}$ from the relation $\boldsymbol{\tau} = \mathbf{F}\mathbf{T}\mathbf{F}^T$, where $\mathbf{T}$ denotes second Piola–Kirchhoff stress, $\mathbf{F}$ is deformation gradient and $\mathbf{L}$ is the Eulerian deformation-rate.

Incompressible "upper-convected Oldroyd" fluid has constitutive relation

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}^d; \quad \frac{\delta\boldsymbol{\sigma}^d}{\delta t} + \frac{\boldsymbol{\sigma}^d}{\tau} = \frac{2\mu}{\tau}\mathbf{D} + 2\mu_r\frac{\delta\mathbf{D}}{\delta t},$$

where $\mathbf{D}$ denotes the Eulerian strain-rate and $\tau$, $\mu$ and $\mu_r$ are positive constants.. Give this relation explicitly, in either matrix or component form, for the case of the pure shear deformation $\mathbf{X} \to \mathbf{x}$:

$$x_1 = X_1 + f(X_2, t),$$
$$x_2 = X_2,$$
$$x_3 = X_3$$

and show that, in the case of steady motion, so that $\dot{\gamma} \equiv \partial^2 x_1/\partial X_2 \partial t$ is independent of $t$,

$$\sigma_{11} = -p + 2(\mu - \mu_r)\dot{\gamma}^2\tau, \quad \sigma_{12} = \mu\dot{\gamma}, \quad \sigma_{22} = \sigma_{33} = -p, \quad \sigma_{13} = \sigma_{23} = 0. \qquad (*)$$

Steady Couette flow between differentially-rotating cylinders has the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{v(r)}{r}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad a < r < b,$$

where $r = (x_1^2 + x_2^2)^{1/2}$. Calculate $\mathbf{L}$ for this flow. Explain why, relative to polar coordinates $(r, \theta)$, the associated stress components conform to the relation $(*)$, with $\dot{\gamma}$ suitably defined. Give $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ explicitly.

In the absence of any body-force and neglecting inertia, find the pressure explicitly (up to an unknown constant) in terms of the speed $v(a)$ of the inner boundary, given that the outer boundary is stationary.

[*The equations of motion (neglecting inertia) are*

$$d(r^2\sigma_{r\theta})/dr = 0, \; d\sigma_{rr}/dr + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0.]$$

**6** An incompressible non-hardening anisotropic plastic material, in plane strain deformation, has yield criterion

$$f(\xi, \tau) = 0; \;\; \xi = \frac{\sigma_{11} - \sigma_{22}}{2}, \;\; \tau = \sigma_{12},$$

and it conforms to the associated flow law

$$D_{ij} = \dot{\lambda} \partial f / \partial \sigma_{ij}.$$

Expressing the yield criterion in the $(\xi, \tau)$ plane in the form $\xi = \xi(l)$, $\tau = \tau(l)$, where $l$ denotes arc length, let

$$d\xi/dl = -\cos(2\phi), \;\; d\tau/dl = -\sin(2\phi)$$

(so that the outward normal to the yield curve makes an angle $-2\phi$ to the $\tau$-axis). Define also $\sigma = (\sigma_{11} + \sigma_{22})/2$. Assuming yield, express the equations of equilibrium in terms of $\sigma$ and $l$.

By considering $dF(\sigma, l)/ds$ along a curve defined parametrically by $(x_1(s), x_2(s))$, show that $F$ is constant along the curve, provided

$$(F_\sigma \;\; F_l) \begin{pmatrix} x_1' \cos(2\phi) + x_2' \sin(2\phi) & x_1' \sin(2\phi) - x_2' \cos(2\phi) \\ x_1' & x_2' \end{pmatrix} = (0 \;\; 0).$$

Deduce that

$$\sigma - l = \text{constant on an } \alpha\text{-line: } dx_2/dx_1 = \tan\phi,$$
$$\sigma + l = \text{constant on a } \beta\text{-line: } dx_2/dx_1 = -\cot\phi.$$

By locally choosing axes so that the $x_1$-axis is tangent to the $\alpha$-line, deduce from the flow law that

$$du - v d\phi = 0 \;\; \text{along an } \alpha\text{-line},$$
$$dv + u d\phi = 0 \;\; \text{along a } \beta\text{-line},$$

where $(u, v)$ denote the components of velocity along the $\alpha$- and $\beta$-lines.

## END OF PAPER