UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 1.30 to 3.30

PAPER 75

MAGNETOHYDRODYNAMICS AND TURBULENCE

Attempt ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

The exam is closed-book, but it is allowed to use vector identities from NRL Plasma Formulary, p. 5 (supplied)

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
NRL Plasma Formulary, p. 5

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Ambipolar Damping in Partially Ionised Plasma. Consider the following incompressible two-fluid model for a plasma with a neutral component:

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{\nabla p}{\rho_i} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi\rho_i} - \mu_{in}\left(\mathbf{u}_i - \mathbf{u}_n\right), \quad \nabla\cdot\mathbf{u}_i = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{u}_n = -\frac{\nabla p_n}{\rho_n} - \mu_{ni}\left(\mathbf{u}_n - \mathbf{u}_i\right), \quad \nabla\cdot\mathbf{u}_n = 0, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u}_i, \tag{3}$$

where $\mathbf{u}_i$ and $\mathbf{u}_n$ are the velocities of the ions and of the neutral particles, respectively; all diffusion terms have been neglected; μ_{in} is the ion-neutral collision rate, $\mu_{ni} = (\rho_i/\rho_n)\mu_{in}$ is the neutral-ion collision rate. For simplicity, assume $\rho_i = \rho_n$.

(a) Consider linear perturbations about a static equilibrium with a uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. Write the ion and neutral velocities in terms of ion and neutral displacements and derive the dispersion relation. It should be a cubic equation in ω.

(b) Assume $k_\parallel v_A \ll \mu_{in}$, where $k_\parallel$ is the wave number in the z direction and $v_A = B_0/\sqrt{4\pi\rho_i}$ is the Alfvén speed. Find approximate expressions for all three solutions of the dispersion relation. One is a pure damping; the other two are weakly damped Alfvén waves (their damping is called *ambipolar damping*).

[*Hint. To obtain the ambipolar damping solve the dispersion relation by successive approximations.*]

(c) Find the relationship between the ion and neutral displacements for each of these solutions. Show in particular that for the ambipolar-damped Alfvén waves, there is a small slippage of ions relative to the neutrals.

2 Reduced Electron MHD. Interstellar and solar-wind turbulence at scales smaller than the ion inertial scale $d_i = c(m_i/4\pi e^2 n)^{1/2}$ can be described by an approximation in which the magnetic field is frozen into the electron flow $\mathbf{u}_e$, while the ions can be considered motionless, $\mathbf{u}_i = 0$. In this approximation, the magnetic field obeys

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n}\nabla\times[(\nabla\times\mathbf{B})\times\mathbf{B}]\,. \tag{4}$$

This is called "Electron MHD" (EMHD). Consider a static equilibrium with a straight uniform magnetic field in the z direction, so that $\mathbf{B} = B_0\hat{\mathbf{z}} + \delta\mathbf{B}$.

Infinitesimal perturbations in this system are linear waves ("Kinetic Alfvén Waves," or KAW) with the dispersion relation

$$\omega(\mathbf{k}) = \pm k_\parallel v_A k d_i, \tag{5}$$

where $v_A = B_0/\sqrt{4\pi n m_i}$ and $k = |\mathbf{k}|$.

Now consider perturbations that are small, but not infinitesimally so. In a way similar to the derivation of the Reduced MHD equations, assume that the perturbations are highly anisotropic, so a small parameter $\epsilon \sim k_\parallel/k_\perp \ll 1$ can be introduced and a reduced version of EMHD derived. Assume further that the wave frequency and the nonlinear interaction time are same order — the critical balance assumption.

(a) Use the critical balance assumption to order the size of the perturbations: estimate the nonlinear interaction time and show that the critical balance assumption implies

$$\frac{\delta B}{B_0} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon. \tag{6}$$

This is the ordering that allows one to derive the Reduced Electron MHD.

(b) Show that the magnetic field can be represented as follows:

$$\frac{\delta\mathbf{B}}{B_0} = \frac{1}{v_A}\,\hat{\mathbf{z}}\times\nabla_\perp\Psi + \hat{\mathbf{z}}\,\frac{\delta B_\parallel}{B_0}. \tag{7}$$

(c) Show that the evolution equations for Ψ and $\delta B_\parallel$ are

$$\frac{\partial\Psi}{\partial t} = v_A^2 d_i\,\frac{\mathbf{B}}{B_0}\cdot\nabla\frac{\delta B_\parallel}{B_0}, \tag{8}$$

$$\frac{\partial}{\partial t}\frac{\delta B_\parallel}{B_0} = -d_i\,\frac{\mathbf{B}}{B_0}\cdot\nabla\nabla_\perp^2\Psi, \tag{9}$$

where

$$\frac{\mathbf{B}}{B_0}\cdot\nabla = \frac{\partial}{\partial z} + \frac{\delta\mathbf{B}_\perp}{B_0}\cdot\nabla_\perp = \frac{\partial}{\partial z} + \frac{1}{v_A}\{\Psi,\cdots\}\,. \tag{10}$$

(d) Check that these equations, when linearised, give the dispersion relation (5) for KAW.

3 Goldreich–Sridhar Turbulence.

(a) Outline the argument that leads to the $k_\perp^{-5/3}$ spectrum for strong Alfvénic turbulence. State all assumptions clearly.

(b) Explain what happens to fluctuations that are polarised as slow and entropy modes.

(c) How many conserved quantities are there in anisotropic strong MHD turbulence? What are they? What is the relationship between the cascades of these quantities?

END OF PAPER