UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 1.30 to 3.30

PAPER 70

ACCRETION DISCS

*There are **THREE** questions in total.*

The questions carry equal weight.

*Full marks can be obtained by completing **TWO** questions.*

*This is an **OPEN BOOK** examination.*

Candidates may bring their course handouts, including example sheets, and any handwritten material into the examination. Textbooks, laptops etc are not permitted.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Paczynski-Wiita potential

$$\Phi(r) = -\frac{GM}{r - R_s},$$

valid for spherical radius $r > R_s = 2GM/c^2$, was chosen to provide a Newtonian analogue for some aspects of the Schwarzschild metric for a black hole of mass M. Compute the epicyclic frequency $\kappa(R)$ and deduce that an accretion disc in such a potential has its inner radius at $R = 3R_s$.

By choosing an appropriate boundary condition at $R = 3R_s$, show that a steady disc, with mass accretion rate $\dot{M}$, accretes angular momentum at a rate

$$C = \frac{3\sqrt{3}}{2}\dot{M}\left(GMR_s\right)^{\frac{1}{2}}.$$

Show that the energy release per unit area for such a disc is given by

$$D(R) = \frac{GM\dot{M}}{4\pi}\frac{(3R - R_s)}{R\,(R - R_s)^3}\left\{1 - \frac{3\sqrt{3}}{2}\left(\frac{R_s}{R}\right)^{\frac{1}{2}}\frac{R - R_s}{R}\right\}.$$

Use this result to compute the total disc luminosity in the form

$$L = \epsilon \dot{M} c^2,$$

where ϵ is to be determined.

Verify your answer by computing directly the energy lost by a disc particle moving from $R = \infty$ to $R = 3R_s$.

[You may assume that for $n = 1$

$$\int \frac{dx}{x^{3/2}(x-a)^n} = \frac{2}{ax^{1/2}} + \frac{1}{a^{3/2}}\ln\left(\frac{x^{1/2} - a^{1/2}}{x^{1/2} + a^{1/2}}\right),$$

and that the results for integers $n > 1$ may be obtained by differentiating with respect to a.]

2 An accretion disc around a point mass, M, is accreting steadily at a rate $\dot{M}$. The surface density has the form $\Sigma = \Sigma_0 R^{-2}$, where Σ_0 is a constant. Show that, far from the inner edge, this implies that the viscosity has the form $\nu = \nu_0 R^2$, where $\nu_0 \Sigma_0 = \dot{M}/3\pi$.

Show that far from the inner edge the radial velocity is $v_R = -3\nu/2R$.

The fluid in the disc contains a trace contaminant with surface density $\sigma \ll \Sigma$, and with concentration $C \equiv \sigma/\Sigma \ll 1$. The quantity σ obeys the equation

$$\frac{\partial \sigma}{\partial t} + \operatorname{div} \mathbf{F} = 0,$$

where the flux of contaminant, $\mathbf{F}$ is given by

$$\mathbf{F} = \sigma \mathbf{u} - k\Sigma \nabla C.$$

Here $\mathbf{u}$ is the ambient velocity field and k a diffusion coefficient. Show that C satisfies the equation

$$\Sigma \left[\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)\, C \right] = \operatorname{div}\, (k\Sigma \nabla C).$$

Explain briefly why in an accretion disc we may expect C just to be a function of R and t.

If $k = \zeta \nu$, where ζ is a constant, show in the steady disc discussed above, far from the inner edge, the equation for C becomes

$$\frac{1}{\nu_0} \frac{\partial C}{\partial t} - \frac{3}{2} \frac{\partial C}{\partial Y} = \zeta \frac{\partial^2 C}{\partial Y^2},$$

where $Y = \ln R$.

At time $t = 0$, a quantity of contaminant is introduced at radius R_0, so that

$$C(R, t = 0) = C_0 \delta\, (R - R_0)\,,$$

where C_0 is a constant. By setting $X = Y + \frac{3}{2}\nu_0 t$, or otherwise, show that at times $t > 0$,

$$C(R, t) = \frac{C_0}{R_0}\, (4\pi \zeta \nu_0 t)^{-1/2} \exp \left\{ - \frac{\left[\ln (R/R_0) + \frac{3}{2}\nu_0 t \right]^2}{4\zeta \nu_0 t} \right\}.$$

Define $F(r, t)$ as the fraction of contaminant which is at radii $R > rR_0$ at time t. Show that

$$F(r, t) = \frac{1}{2} \operatorname{erfc} \left\{ \frac{\left(\ln r + \frac{3}{2}\nu_0 t\right)}{\sqrt{4\zeta \nu_0 t}} \right\}.$$

Deduce that the probability that an element of contaminant, which starts at R_0, reaches a radius $R_1 = r_1 R_0$ is given by

$$\begin{aligned} P\,(r_1) &= \max_{0<t<\infty} F\,(r_1, t) \\ &= \frac{1}{2} \operatorname{erfc} \left\{ (3 \ln r_1 / 2\zeta)^{\frac{1}{2}} \right\}. \end{aligned}$$

[You may assume that the general solution of the equation $u_{xx} - u_t = 0$ is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(x,0) \exp\left\{ -\frac{(x-\xi)^2}{4t} \right\} d\xi,$$

and that

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2}\, dt.\Big]$$

3 A centrifugally supported, thin disc lies in the symmetry plane ($z = 0$) of a gravitational potential of the form $\Phi(R,z) = \Phi(R,-z)$. Let $\Omega(R)$, $\kappa(R)$ and $\Omega_z(R)$ be the orbital angular frequency, the epicyclic frequency and the vertical oscillation frequency associated with nearly circular orbits at radius R. If the disc is locally isothermal with sound speed $c_s(R)$ show that the vertically integrated pressure $P(R) \equiv \int_{-\infty}^{\infty} p\,(R,z)\,dz$ is given (approximately for a thin disc) by

$$P = \Omega_z^2 \Sigma H^2,$$

where Σ is the surface density, and $H = c_s/\Omega_z$ the local density scale-height.

For a nearly Keplerian disc, explain briefly the physical meaning of the two frequencies

$$\omega_1 = \left(\Omega^2 - \Omega_z^2\right)/2\Omega$$

and

$$\omega_2 = \left(\Omega^2 - \kappa^2\right)/2\Omega\,.$$

The equations governing small amplitude warping disturbances of a nearly Keplerian, inviscid disc can be written in the form

$$\Sigma R^2 \Omega \left[\frac{\partial W}{\partial t} - i\omega_1 W\right] = \frac{1}{R}\frac{\partial G}{\partial R}\,,$$

and

$$\frac{\partial G}{\partial t} - i\omega_2 G = \frac{1}{4} P R^3 \Omega \frac{\partial W}{\partial R}\,.$$

Here $W(R,t) = \beta(R,t)\exp[i\gamma(r,t)]$ represents the (small) tilt at each radius, with tilt angle β at azimuthal angle γ, and G represents an internal torque in the disc plane.

In an exactly Keplerian disc, around point mass M, show that such disturbances with wavelengths $\lambda \ll R$, propagate as warp waves with speed $\frac{1}{2}c_s$.

Around a Kerr black hole, with spin parameter $a\,(-1 \leqslant a \leqslant 1)$ the following relationships approximately hold:

$$\begin{aligned}\Omega^{-1} &= \left(\frac{GM}{c^3}\right) r^{3/2},\\ \omega_1 &= 2a\Omega r^{-3/2},\\ \omega_2 &= 3\Omega r^{-1},\end{aligned}$$

where $r = Rc^2/GM$. A disc around a Kerr black hole is tilted at large radii at a fixed angle W_∞. At the inner disc edge $r = r_{\rm in}$ the internal disc torque vanishes ($G = 0$). Show that if the hole is not spinning ($a = 0$) the disc can remain flat (so that $W = W_\infty$ for all radii, $r_{\rm in} \leqslant r \leqslant \infty$).

If $a \neq 0$, and if $a \ll r^{3/2}$ so that we may assume $P = \Omega^2\Sigma H^2$, show that the disc can take on a steady warped shape $W(R)$, where W satisfies the equation

$$\frac{d}{dR}\left[\Sigma H^2 r \frac{dW}{dR}\right] + 24 a r^{-3/2} \Sigma W = 0\,.$$

Find, and describe, the shape of the warp in the case when $\Sigma \propto R^{-3/4}$, $H/R = \epsilon =$ constant, and

$$r_{\text{in}}^{5/4} = \frac{4\sqrt{24a}}{5\pi\epsilon}\,.$$

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