UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 1.30 to 4.30

PAPER 7

TOPICS IN GROUP THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a finite group.

(i) Define in terms of series what it means for G to be *nilpotent*.

(ii) Prove that any p-group (for p a prime) is nilpotent. [You may use without proof the result that any non-trivial p-group has non-trivial centre.]

(iii) Prove that if $K \trianglelefteq H \leqslant G$, then $[H, G] \leqslant K$ if and only if $K \trianglelefteq G$ and $H/K \leqslant Z(G/K)$.

(iv) Prove that a direct product of two nilpotent groups is also nilpotent.

(v) Define the *lower central series*

$$G = \Gamma_1(G) \trianglerighteq \Gamma_2(G) \trianglerighteq \Gamma_3(G) \trianglerighteq \cdots$$

of G and the *upper central series*

$$1 = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \cdots$$

of G. Show that G is nilpotent if and only if $\Gamma_n(G) = 1$ for some n if and only if $Z_n(G) = G$ for some n. Moreover, suppose G is nilpotent and

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$$

is any central series of G; show that for $i = 0, \ldots, r$ we have $\Gamma_{r-i+1}(G) \leqslant G_i \leqslant Z_i(G)$, and deduce that for $c > 0$ we have $\Gamma_{c+1}(G) = 1$ if and only if $Z_c(G) = G$.

(vi) Let G be a Sylow 2-subgroup of S_7. Give generators for G, and find its lower and upper central series.

2 Let (G, Ω) be a transitive permutation group, where Ω is a finite set.

(i) Define what it means for a permutation group (G^+, Ω^+) to be a *one-point extension* of (G, Ω).

(ii) Show that the permutation group $(D_{2(p-1)}, \Omega)$, where $p \geqslant 5$ is prime and $\Omega = \{1, \ldots, p-1\}$, has no one-point extension.

(iii) State and prove necessary and sufficient conditions for the existence of a one-point extension of (G, Ω).

(iv) Show that, for each $n \geqslant 3$, the alternating group A_n in its natural action satisfies the conditions in (iii) and therefore has a one-point extension.

3 Let V be a vector space of dimension n over the field of q elements.

(i) Define what it means for a linear map in $\mathrm{GL}(V)$ to be a *transvection*. Show that any transvection may be written as $\tau_{f,d} : v \mapsto v + (vf)d$ for $v \in V$, where f is a linear functional on V and $0 \neq d \in \ker f$. Calculate the product of $\tau_{f,d}$ and $\tau_{f',d}$, and the result of conjugating $\tau_{f,d}$ by an element g of $\mathrm{GL}(V)$.

Let $\mathcal{T}$ denote the set of transvections in $\mathrm{GL}(V)$, and $\mathcal{T}^{\#} = \mathcal{T} \setminus \{1\}$.

(ii) Prove that if $n \geqslant 2$ then $\mathcal{T}^{\#}$ forms a single conjugacy class in $\mathrm{GL}(V)$, and lies in $\mathrm{SL}(V)$; prove moreover that if $n \geqslant 3$ then $\mathcal{T}^{\#}$ forms a single conjugacy class in $\mathrm{SL}(V)$.

(iii) Prove that $\mathcal{T}$ generates $\mathrm{SL}(V)$.

(iv) Prove that if $n \geqslant 2$, and $(n, q) \neq (2, 2)$ or $(2, 3)$, then $\mathrm{SL}(V)$ is perfect.

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(i) Show that if $n \neq 6$ then $\mathrm{Out}\, S_n = 1$, while $|\mathrm{Out}\, S_6| \leqslant 2$.

(ii) Show that S_6 has exactly 6 (synthematic) totals.

Label the totals of S_6 as shown:

$$\begin{aligned}
T_1 &= \{(1\,2)(3\,4)(5\,6), (1\,3)(2\,5)(4\,6), (1\,4)(2\,6)(3\,5), (1\,5)(2\,4)(3\,6), (1\,6)(2\,3)(4\,5)\},\\
T_2 &= \{(1\,2)(3\,4)(5\,6), (1\,3)(2\,6)(4\,5), (1\,4)(2\,5)(3\,6), (1\,5)(2\,3)(4\,6), (1\,6)(2\,4)(3\,5)\},\\
T_3 &= \{(1\,2)(3\,6)(4\,5), (1\,3)(2\,5)(4\,6), (1\,4)(2\,3)(5\,6), (1\,5)(2\,6)(3\,4), (1\,6)(2\,4)(3\,5)\},\\
T_4 &= \{(1\,2)(3\,6)(4\,5), (1\,3)(2\,4)(5\,6), (1\,4)(2\,6)(3\,5), (1\,5)(2\,3)(4\,6), (1\,6)(2\,5)(3\,4)\},\\
T_5 &= \{(1\,2)(3\,5)(4\,6), (1\,3)(2\,6)(4\,5), (1\,4)(2\,3)(5\,6), (1\,5)(2\,4)(3\,6), (1\,6)(2\,5)(3\,4)\},\\
T_6 &= \{(1\,2)(3\,5)(4\,6), (1\,3)(2\,4)(5\,6), (1\,4)(2\,5)(3\,6), (1\,5)(2\,6)(3\,4), (1\,6)(2\,3)(4\,5)\}.
\end{aligned}$$

The set $\{T_1, \ldots, T_6\}$ has a natural action of S_6 by conjugation; define $\alpha \in \mathrm{Aut}\, S_6 \setminus \mathrm{Inn}\, S_6$ by $T_i{}^{\pi} = T_{i\pi^{\alpha}}$ for $\pi \in S_6$ and $i = 1, \ldots, 6$.

(iii) Give the image under α of the transpositions $(1\,2)$, $(2\,3)$, $(3\,4)$, $(4\,5)$ and $(5\,6)$. By taking suitable products of these, identify two elements of S_6 of order 2 which are fixed by α; hence find a 5-cycle fixed by α.

(iv) Define what is meant by a *Steiner system* $S(5, 6, 12)$. Explain how to obtain such a Steiner system on the set $\{1, \ldots, 6, T_1, \ldots, T_6\}$.

END OF PAPER