UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2007 9.00 to 12.00

PAPER 69

ASTROPHYSICAL DYNAMICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the term *integral of motion*. Show that in a steady-state spherical system the binding energy E and total angular momentum $\mathbf{L}$

$$E = \psi(r) - \tfrac{1}{2}v^2, \qquad \mathbf{L} = \mathbf{r} \times \mathbf{v}$$

are integrals of the motion. Show that the distribution function of a collisionless steady-state stellar system depends on the integrals of motion only (*Jeans Theorem*).

Consider a distribution function of form

$$f(E, L) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{E}{\sigma^2}\right) \exp\left(-\tfrac{1}{2}\left[\frac{L}{r_a\sigma}\right]^2\right),$$

where σ and r_a are constants, and $\mathrm{L} = |\mathbf{L}|$. By introducing spherical polar coordinates in velocity space, show that the density is

$$\rho = \frac{\rho(0)}{(1 + r^2/r_a^2)} \exp\left[\frac{\psi(r) - \psi(0)}{\sigma^2}\right]$$

and deduce that the velocity second moments are $\langle v_r^2 \rangle = \sigma^2$ and $\langle v_\phi^2 \rangle = \langle v_\theta^2 \rangle = \sigma^2 r_a^2/(r_a^2 + r^2)$

Show that for $r << r_a$, the density $\rho \propto r^{-2}$, while for $r >> r_a$, $\rho \propto r^{-2}(\log r)^{-1}$. What is the corresponding behaviour of the circular speed in these two regimes? Give a physical interpretation of the structure of the model.

Hint: You may assume the following integrals without proof

$$\int_0^\pi \frac{\sin\eta d\eta}{(1 + k^2\sin^2\eta)^{3/2}} = \frac{2}{1 + k^2}$$

$$\int_0^\pi \frac{\sin\eta\cos^2\eta d\eta}{(1 + k^2\sin^2\eta)^{5/2}} = \frac{2}{3(1 + k^2)}$$

$$\int_0^\infty x^4 \exp(-k^2x^2) dx = \frac{3\sqrt{\pi}}{8k^5}$$

2 (a) State *Gauss' Theorem* and use it to find the axisymmetric density corresponding to the gravitational potential

$$\phi(r,\theta) = v_0^2 \log(r(1 + |\cos\theta|)),$$

where v_0 is a constant. What is the circular velocity curve $v_{\rm circ}$ of the model? What is the mass $M(r)$ enclosed within radius r? Compare $v_{\rm circ}^2$ to $GM(r)/r$ and comment on your result.

(b) An infinitesimally thin disk lies in the plane $z = 0$ and has surface density $\Sigma(x, y)$ and gravitational potential $\phi(x, y)$. By using Fourier transforms, or otherwise, derive the following two relations

$$\Sigma(\mathbf{x}) = \frac{1}{4\pi^2 G} \int\int \frac{dx'dy'}{|\mathbf{x} - \mathbf{x}'|}\Big(\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2}\Big)$$

$$\phi(\mathbf{x}) = -G \int\int dx'dy' \frac{\Sigma(x', y')}{|\mathbf{x} - \mathbf{x}'|}$$

Hint: You may assume the following Fourier transform without proof

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp i(kx + \ell y) \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{k^2 + \ell^2}}$$

as well as any standard theorems in Fourier analysis.

3 Using a clear diagram, define the *orbital elements* of a Keplerian ellipse, namely the semimajor axis a, the eccentricity e, the longitude of the ascending node Ω, the argument of pericenter ω and the inclination i.

The Hamiltonian of the Kepler problem can be written as

$$H_K(J_r, J_\theta, J_\phi) = -\frac{G^2M^2}{2(J_r + J_\theta + J_\phi)^2}$$

where J_r, J_θ and J_ϕ are the actions referred to spherical polar coordinates. Find a canonical transformation to a new set of actions (J_1, J_2, J_3) and angles (w_1, w_2, w_3) such that H becomes a function of J_1 alone. Explain how the new actions and angles (the *Delaunay variables*) are related to the orbital elements.

Consider the case of a test particle orbiting very close to a planet with mass M. Let the gravitational potential consist of a monopole term and a quadrupole describing the planet's flattening

$$\phi = -\frac{GM}{r} + \alpha^2 GM\frac{(3z^2 - r^2)}{r^5},$$

where α is a constant. Find the radial and vertical frequencies of motion. Show that the the line of nodes of the test particle precesses backwards, while the perihelion position precesses forwards.

4 Show from the virial theorem that a self-gravitating system has negative heat capacity. Explain what is meant by the *gravothermal catastrophe.*

Consider a system of N self-gravitating infinite rods of equal mass per unit length m confined to an infinite cylinder of radius R_e. The rods are restricted to move in two dimensions, with their axes parallel to that of the cylinder. Let the gravitational potential be normalised so that the potential at the boundary is

$$\psi(R_e) = -GM \log(V/V_0)$$

where M $= Nm$, $V = \pi R_e^2$ and $V_0 = \pi R_0^2$. Anywhere else between the axis of the cylinder and the boundary, the potential $\psi(R)$ must satisfy Poisson's equation

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\psi}{dR}\right) = -4\pi G\rho.$$

Assume that the distribution of the velocities of the rods is of Maxwellian form

$$f \propto \exp\Big[\alpha - \beta(v^2/2 - \psi)\Big]$$

where α is a constant and $\beta = m/(kT)$. Verify that there exist regular solutions of the form

$$\psi(R) = -GM \log(V/V_0) - 2\beta^{-1} \log\left[1 - \tfrac{1}{2}GM\beta(1 - R^2/R_e^2)\right].$$

Hence, show that there exists a lower bound to the temperature $T_{\min}$ given by

$$kT_{\min} = \tfrac{1}{2}GMm\,.$$

Show that the total energy E of the system is

$$E = NkT_{\min}\left[\log(V/V_0) + 2\Theta + \Theta^2 \log(1 - \Theta^{-1})\right]$$

where $\Theta = T/T_{\min}$. Use this expression to demonstrate that there is no gravothermal catastrophe in two dimensions.

END OF PAPER