UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 9.00 to 12.00

PAPER 68

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a Parker wave dynamo model for a mean magnetic field $\mathbf{B} = B(x,t)\mathbf{e}_\phi + \nabla \times A(x,t)\mathbf{e}_\phi$ in the form

$$\frac{\partial A}{\partial t} = \alpha B + \frac{\partial^2 A}{\partial x^2} - \ell^2 A\,, \quad \frac{\partial B}{\partial t} = \Omega\frac{\partial A}{\partial x} + \frac{\partial^2 B}{\partial x^2} - \ell^2 B\,.$$

Here Ω is a constant, but $\alpha = \epsilon^{-1}\tilde{\alpha} = \epsilon^{-1}\alpha_0 \sin\omega\tau$, $\tau = t/\epsilon$, ϵ is a small parameter, and α_0 is a constant.

Defining $A = \overline{A} + \tilde{A}(\tau)$, $B = \overline{B} + \epsilon\tilde{B}(\tau)$, where the overbar indicates an average over the fast (τ) timescale, and $\tilde{A}$, $\tilde{B}$ have zero average, show that the equations for $\overline{A}$, $\overline{B}$ take the form

$$\frac{\partial \overline{A}}{\partial t} = \overline{\tilde{\alpha}\tilde{B}} + \frac{\partial^2 \overline{A}}{\partial x^2} - \ell^2 \overline{A}\,, \quad \frac{\partial \overline{B}}{\partial t} = \Omega\frac{\partial \overline{A}}{\partial x} + \frac{\partial^2 \overline{B}}{\partial x^2} - \ell^2 \overline{B}\,,$$

where $\overline{\tilde{\alpha}\tilde{B}} = -C\partial\overline{B}/\partial x$, and the constant C is to be determined.

Look for solutions of these equations proportional to $e^{ikx+\sigma t}$, and show that growing solutions exist if ΩC is sufficiently large. Give a physical explanation why σ is real, so that the dynamo waves do not travel.

2 Diffusionless waves, describing disturbances to a uniform magnetic field $\mathbf{B}$ in a fluid rotating with angular velocity $\Omega\hat{\mathbf{z}}$, in the presence of gravity $-g\hat{\mathbf{z}}$ and a uniform temperature gradient $-\beta$ in the z-direction, are described in the Boussinesq limit by the equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + 2\Omega\hat{\mathbf{z}} \times \mathbf{u}\right) = -\nabla p + (\mu_0)^{-1}\mathbf{B}\cdot\nabla\mathbf{b} + \alpha\rho\theta g\hat{\mathbf{z}}\,,$$

$$\frac{\partial \theta}{\partial t} = \beta u_z\,,$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}\cdot\nabla\mathbf{u}\,,$$

together with $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0$.

(i) Seek solutions in which all the variables are proportional to $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, and show, by taking the curl of the momentum equation to eliminate p, and eliminating all variables in favour of u_z, $\zeta = (\nabla\times\mathbf{u})_z$ and θ, or otherwise, that the dispersion relation giving ω in terms of $\mathbf{k}$ and the other parameters takes the form

$$\Delta^2|\mathbf{k}|^2 - 4\Omega^2 k_z^2 + \frac{g\alpha\beta}{\omega}\Delta(|\mathbf{k}|^2 - k_z^2) = 0, \quad \text{where} \;\; \Delta = \omega - \frac{(\mathbf{B}\cdot\mathbf{k})^2}{\mu_0\rho\omega}\,.$$

(ii) Now suppose that $|\Omega|$ is very large, and that $k_z^2 \sim k_y^2 \sim k_x^2$. Show that the dispersion relation has two types of solution: one in which $|\omega| = O(|\Omega|)$ and another in which $|\omega| \ll |\Omega|$. In the latter case give the leading order approximation to ω, and thus determine conditions on the parameters so that ω is real.

3 A model of parity interactions in mean field dynamo models consists of the coupled system for complex amplitudes A_1, B_1 of fields in one hemisphere and A_2, B_2 of fields in the other:

$$\frac{\partial A_1}{\partial t} = \alpha_1 B_1 + \epsilon_1 B_2 - A_1, \quad \frac{\partial B_1}{\partial t} = i\Omega A_1 - \epsilon_2 A_2 - B_1,$$
$$\frac{\partial A_2}{\partial t} = -\alpha_2 B_2 - \epsilon_1 B_1 - A_2, \quad \frac{\partial B_2}{\partial t} = -i\Omega A_2 + \epsilon_2 A_1 - B_2.$$

The parameters ϵ_1, ϵ_2 couple the behaviours in the two hemispheres.

(i) Initially $\alpha_1 = \alpha_2 = \alpha_0$(const.), so that the system is linear. Show that the equations admit dipole solutions with $A_1 = A_2$, $B_1 = -B_2$, and quadrupole solutions with $A_1 = -A_2$, $B_1 = B_2$. Seek marginally stable solutions in each case in which all the amplitudes are proportional to $\exp(i\omega t)$ for some real ω and find conditions on the parameters that determine which of these two types of solution becomes unstable at the smaller value of $|\alpha_0 \Omega|$.

(ii) Now put $\epsilon_2 = 0$ and consider a model of α-quenching in which $\alpha_1 = \alpha_0 - p|B_1|^2$, $\alpha_2 = \alpha_0 - p|B_2|^2$, where p is a positive real constant. Show that dipole and quadrupole solutions proportional to $\exp(i\omega t)$ (where ω is to be determined) still exist if $|\alpha_0 \Omega|$ is sufficiently large, and give an expression for $|B_1|^2 = |B_2|^2$ in this case for each parity. Give also conditions on the parameters that permit solutions with the same simple time-dependence for which $|A_1| \neq |A_2|$, $|B_1| \neq |B_2|$. How should such solutions be interpreted physically? [*Hint: Choose* $\omega = 1$ *and set* B_1, B_2 *to be real.*]

4 Write an essay on magnetoconvection. Your essay should cover the effects of an imposed magnetic field on scale selection, the occurrence of oscillatory motion, and subcritical convection at large R_m.

END OF PAPER