UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 1.30 to 4.30

PAPER 67

PHYSICAL COSMOLOGY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Explain the difference between comoving and physical (proper) coordinates. Which subtends a larger angle on the sky at redshift $z = 3$: a transverse distance of 5 proper Mpc or of 5 comoving Mpc? In the small angle approximation, what is the factor which relates the two angular sizes?

(ii) From the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho + \frac{\Lambda}{3}$$

show that, in a Friedmann-Robertson-Walker universe, the scale factor evolves with time according to the equation:

$$\dot{a}^2 = H_0^2 a^2 \left(\Omega_{\mathrm{m},0}\frac{a_0^3}{a^3} + \Omega_{\mathrm{k},0}\frac{a_0^2}{a^2} + \Omega_{\Lambda,0}\right) ,$$

where H is the Hubble parameter, Ω_{m}, Ω_{k}, and Ω_Λ are the fractions of the critical density contributed, respectively, by matter, curvature and the cosmological constant, and the subscript 0 denotes the present time. You may use natural units, whereby $c = 1$, and assume that radiation makes an unimportant contribution to the mass-energy density, i.e. $\Omega_{\mathrm{rad}} = 0$.

(iii) Hence show that in a universe with a positive cosmological constant, the expansion rate $\dot{a}$ reaches a minimum when the scale factor attains the value:

$$a_{\mathrm{min}} = \left(\frac{\Omega_{\mathrm{m},0}}{2\Omega_{\Lambda,0}}\right)^{1/3} a_0 \,.$$

(iv) Derive an expression for $\dot{a}$ when $a = a_{\mathrm{min}}$ in terms of the parameters $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{k},0}$, $\Omega_{\Lambda,0}$, H_0, and a_{min}.

(v) In a flat universe, what is the value of the parameter $E(z) \equiv [H(z)/H_0]^2$ at the redshift z where $a = a_{\mathrm{min}}$?

2 (i) Show that in a flat universe with only matter and a cosmological constant ($\Omega_{\mathrm{k},0} = 0$; $\Omega_{\mathrm{rad},0} = 0$; $\Omega_{\mathrm{m},0} \neq 0$; $\Omega_{\Lambda,0} \neq 0$), the comoving particle horizon at redshift z is given by the expression:

$$r_{\mathrm{h}} = \int_{z'=z}^{z'=\infty} \frac{c}{H_0 \sqrt{\Omega_{\mathrm{m},0}(1+z')^3 + \Omega_{\Lambda,0}}}\, dz'$$

(ii) Solve this integral in the Einstein-de Sitter case ($\Omega_{\mathrm{k},0} = 0$; $\Omega_{\mathrm{m},0} = 1$; $\Omega_{\Lambda,0} = 0$).

(iii) If $\Omega_{\Lambda,0} \geqslant 0$, a universe with the properties described at (i) will continue to expand so that $a \to \infty$ and $z \to -1$. Sketch the $\Omega_{\Lambda,0} = 0$ solution for $r_{\mathrm{h}}(z)$ in the interval $-1 \leqslant z \leqslant 2$. On the same axes, sketch the behaviour of $r_{\mathrm{h}}(z)$ for our own universe, where $\Omega_{\mathrm{m},0} \simeq 0.3$ and $\Omega_{\Lambda,0} \simeq 0.7$ (you do not need to solve the integral for this latter case). Assume that the quantity $H_0\Omega_{\mathrm{m},0}$ takes the same value in both cases. Suggest a physical interpretation for the behaviour of $r_{\mathrm{h}}(z)$ in the $\Omega_{\Lambda,0} > 0$ universe.

(iv) A photon leaves today ($t = t_0$) from the present particle horizon. Assuming an Einstein-de Sitter universe, at what time will the photon arrive at the Earth? Express your answer in terms of the present age of the universe.

3 (i) An intergalactic gas cloud produces a Lyman alpha absorption line at redshift z_{c} in the spectrum of a quasar at emission redshift z_{q}. Derive an expression between the distance d_{cq} of the cloud from the QSO and the redshift difference $\delta z = z_{\mathrm{q}} - z_{\mathrm{c}}$, under the assumption that δz is small ($\delta z \ll z_{\mathrm{q}}$). Give your answer in terms of the density parameters $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$, assuming a flat cosmology with $\Omega_{\mathrm{k},0} = 0$ and $\Omega_{\mathrm{rad},0} = 0$.

(ii) Calculate the approximate distance in h^{-1} Mpc between a Lyman alpha cloud at $z_{\mathrm{c}} = 3.0$ located in front of a quasar at $z_{\mathrm{q}} = 3.1$, in an Einstein-de Sitter cosmology.

(iii) Typically, the emission redshifts of quasars z_{q} have an associated error $\frac{\Delta z_{\mathrm{q}}}{1+z} \cdot c = \pm 1000\,\mathrm{km\ s^{-1}}$, whereas the absorption redshifts of Lyman alpha clouds are much more precisely determined, so that $\Delta z_{\mathrm{c}} \ll \Delta z_{\mathrm{q}}$. What is the resulting uncertainty $\frac{\Delta d_{\mathrm{cq}}}{d_{\mathrm{cq}}}$ in the distance between the Lyman alpha cloud and the quasar in (ii)?

(iv) Explain what is meant by the 'Proximity Effect' and explain how this effect can be used to measure the intensity of the metagalactic ionising background. Your essay should include a description of the Lyman forest and its redshift evolution. Discuss the likely origin of the metagalactic ionizing background.

4 The lens equation for a thin, spherically symmetric, gravitational lens is given by:

$$\beta = \theta - \frac{D_{\rm ds}}{D_{\rm s} D_{\rm d}} \frac{4GM(\theta)}{c^2\theta}\,, \tag{1}$$

where β and θ are, respectively, the source and image positions from the centre in angular coordinates; $D_{\rm ds}$, $D_{\rm s}$ and $D_{\rm d}$ are deflector-source, source and deflector angular diameter distances respectively; $M(\theta)$ is the mass enclosed within θ, and the other symbols have their usual meaning.

A simplified model often used to describe the mass distribution of a galaxy or galaxy cluster is a singular isothermal sphere (SIS), whose projected surface mass density is:

$$\Sigma(\xi) = \frac{\sigma^2}{2\,G\,\xi} \tag{2}$$

where ξ is the physical (proper) distance from the centre and σ is the velocity dispersion.

For the rest of the question you may use: $c = 3 \times 10^8$ m s^{-1}; $1\,{\rm Gpc} = 3 \times 10^{25}$ m; $G = 7 \times 10^{-11}$ N m^2 kg^{-2}; $1\,M_\odot = 2 \times 10^{30}$ kg.

(i) A distant galaxy lies behind a foreground cluster of galaxies and an Einstein ring is produced.

What is the condition for the production of an Einstein ring?

From eq. (1) write an expression for the Einstein radius as a function of the enclosed mass.

If the background galaxy and the foreground cluster are at distances such that the ratio $\frac{D_{\rm ds}}{D_{\rm s} D_{\rm d}} = 1\,{\rm Gpc}^{-1}$, calculate approximately the enclosed mass (in solar masses $M_\odot$) if the Einstein ring has a radius of 1 minute of arc ($\simeq 3 \times 10^{-4}$ radians).

(ii) Assuming that the mass distribution of the lensing cluster can be approximated by a singular isothermal sphere, use eqs. (1) and (2) to derive an expression for the Einstein radius in terms of the velocity dispersion σ and the ratio of angular diameter distances $D_{\rm ds}/D_{\rm s}$.

Hence obtain an approximate estimate for the velocity dispersion of the cluster if $D_{\rm ds}/D_{\rm s} = 0.5$.

(iii) Consider now the case where the background galaxy is not as well aligned with the cluster centre as in (i), but is located 4 minutes of arc from the axis joining the observer and cluster centre. Are multiple images of the galaxy seen by the observer? Give a reason for your answer.

State the relationship between the convergence κ and the deflection angle $\vec{\alpha} = (\beta - \theta)\,(\vec{\theta}/\theta)$, and thereby show that $\kappa = \theta_{\rm E}/2\theta$.

If the galaxy were intrinsically circular, describe qualitatively how it would appear to an observer on Earth.

If the magnitude of the shear is equal to the convergence ($|\gamma| = \kappa$), calculate the magnification factor.

(iv) Describe in a few sentences what is meant by cosmic shear and explain its importance in physical cosmology.

END OF PAPER