UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 9.00 to 12.00

PAPER 66

STRUCTURE AND EVOLUTION OF STARS

Attempt ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

The symbols used in these questions have the meanings they were given in the lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}.$$

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi a c r^2 T^3}.$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho\epsilon.$$

$$P = \frac{\mathcal{R}\rho T}{\mu} + \frac{aT^4}{3},$$

with $1/\mu = 2X + 3Y/4 + Z/2$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In a cluster containing stars with variable but homogeneous composition, the stellar material is an ideal gas with radiation pressure being negligible. Energy production is due to the *pp* chain with energy generation rate per unit mass being given by $\epsilon = \epsilon_0 X^2 \rho T^4$, where ϵ_0 is a constant.

Energy transport is by radiation and the opacity $\kappa = \kappa_0 Z(1+X)\rho T^{-4}$, where κ_0 is constant.

A set of dimensionless variables are defined through $x = r/R$, $q = m/M$, $l = L_r/L$, $b = (4\pi\rho R^3)/M$ and $p = (4\pi R^4 P)/(GM^2)$, with q, l, b, p being functions only of x.

Show that in terms of these variables, the equations of stellar structure take the form

$$\frac{dp}{dx} = -\frac{bq}{x^2},$$

$$\frac{dq}{dx} = x^2 b,$$

$$\frac{d}{dx}\left(\frac{p}{b}\right) = -D\frac{b^9 l}{x^2 p^7},$$

$$\frac{dl}{dx} = E x^2 p^4 b^{-2},$$

where

$$D = \frac{3\kappa_0 Z(1+X)\mathcal{R}^8 LR}{256\pi^3 ac\mu^8 G^8 M^6} \quad \text{and}$$

$$E = \frac{\epsilon_0 X^2 M^6}{4\pi R^7 L}\left(\frac{\mu G}{\mathcal{R}}\right)^4 .$$

Hence show that the stars obey a relation between luminosity, mass and composition of the form

$$L \propto \frac{M^6 \mu^{26/3}}{X^{1/3} Z^{7/6} (1+X)^{7/6}},$$

and find a corresponding relation for the radius.

The stars reach the main sequence with the same value $X = X_0$ but with a range of values of Z. Adopting the approximate relation $\mu = 4/(5X+3)$, show that at that stage, stars with a given luminosity satisfy the relation

$$T_e \propto Z^{-1/12},$$

where T_e is the effective temperature.

The stars evolve converting hydrogen to helium while Z remains fixed. Assuming the chemical composition remains homogeneous, show that

$$\frac{dX}{dt} = -\frac{L}{ME_H},$$

where the constant E_H is the energy released when a unit mass of hydrogen is converted to helium.

A star turns off the main sequence when $X = fX_0$ where f is a fixed constant < 1. Show that after evolving for a time t, stars that are turning off the main sequence obey the relation

$$M \propto Z^{7/30} t^{-1/5}.$$

2 Derive Schwarzschild's condition for stability to convection of a stellar radiative region consisting of an ideal gas with ratio of specific heats $\gamma = 5/3$ in the form

$$\frac{P}{T}\frac{dT}{dP} < \frac{2}{5}.$$

The temperature in the atmosphere of a cool star is given as a function of the optical depth τ by

$$T^4 = T_e^4 \left(\frac{1}{2} + \frac{3}{4}\tau \right) ,$$

and the opacity is given by $\kappa = \kappa_0 \rho T^{4\beta+1}$, where $\beta > 1$ and κ_0 are constants.

Show that in the upper radiative layers

$$P^2 = \frac{2^{\beta+2}\mathcal{R}g}{3\kappa_0(\beta-1)\mu T_e^{4\beta}} \left(1 - \frac{1}{(1+3\tau/2)^{\beta-1}} \right) ,$$

where g is the acceleration due to gravity at the stellar surface.

Deduce that convection sets in when

$$\tau = \frac{2}{3} \left(\left(\frac{4\beta+1}{5} \right)^{1/(\beta-1)} - 1 \right) .$$

In the convective region just below the radius of onset, the structure is polytropic with $P = KT^{5/2}$. Show that the scaling of K with g and T_e is given by

$$K \propto \frac{g^{1/2}}{T_e^{2\beta+5/2}}.$$

In the deeper convective layers the opacity is given by $\kappa = \kappa_1 \rho T^{-7/2}$ where κ_1 is a constant. These layers have negligible mass, no energy generation and there is a transition to an inner radiative zone. Show that this occurs where

$$\frac{3\kappa_1 \mu K^2 L}{16\pi ac\mathcal{R}GMT^{7/2}} = \frac{2}{5} ,$$

and hence that the scaling of the transition temperature with T_e is given by

$$T \propto T_e^{-(8\beta+2)/7}.$$

(You may assume that the same polytropic relation applies throughout the convection zone)

3 (a) Derive the virial theorem for a spherically symmetric star in hydrostatic equilibrium with zero boundary pressure in the form

$$3\int_0^M \frac{P}{\rho} dm + \Omega = 0,$$

where Ω is the gravitational energy.

(b) Show that the electron pressure for a gas consisting of a mixture of hydrogen and helium in which the electrons are completely degenerate but non relativistic is given by the polytropic relation, with index $n = 1.5$,

$$P = K\rho^{5/3},$$

where

$$K = \left(\frac{3}{2\pi}\right)^{2/3} \frac{h^2(1+X)^{5/3}}{40 m_e m_p^{5/3}},$$

with h, m_e, and m_p being Planck's constant, the mass of the electron and the mass of a proton respectively.

When the gas is partially degenerate, the equation of state is approximated by

$$P = K\rho^{5/3} + \left(\frac{\mathcal{R}}{\mu}\right)\rho T.$$

A star consisting of such material contracts towards the main sequence in hydrostatic equilibrium. Assuming the structure remains that of a polytrope with $n = 1.5$, use the virial theorem derived in part (a) to show that the mean temperature is given by

$$\int_0^1 T dq = \frac{2\mu}{7\mathcal{R}}\frac{GM}{R} - \frac{\mu A M^{2/3}}{3\mathcal{R}R^2},$$

where the constant

$$A = 3K\int_0^1 \left(\frac{3\rho}{4\pi\bar{\rho}}\right)^{2/3} dq,$$

with $\bar{\rho} = 3M/(4\pi R^3)$ and $q = m/M$, does not depend on M or R.

Show that during the contraction the mean temperature attains a maximum value given by

$$\int_0^1 T dq = \frac{3\mu G^2 M^{4/3}}{49\mathcal{R}A}.$$

Briefly explain the consequences of this result for stellar structure.

(You may assume that for complete degeneracy, the number density of electrons, $n(p)$, with total momentum less than p, is given by $dn(p)/dp = 8\pi p^2/h^3$ for $p < p_0$ and by $dn(p)/dp = 0$ for $p > p_0$, where p_0 is the Fermi momentum. You may also assume that for a polytrope of index n, $\Omega = -3GM^2/((5-n)R)$ and that $\rho/\bar{\rho}$ depends only on q.)

4 A binary system with components of mass M_1 and M_2 is in circular orbit about the centre of mass and the distance between them is a.

Show that the total angular momentum of the binary is given by

$$J = \left(\frac{M_1 M_2}{M_1 + M_2}\right)(G(M_1 + M_2)a)^{1/2}.$$

The star of mass M_1 transfers mass to M_2 through Roche lobe overflow such that the total mass and angular momentum of the binary are conserved. Show that during this process

$$a(1+q)^{-4}q^2 = C,$$

where $q = M_1/M_2$ and C is a constant.

The Roche lobe radius of M_1 is given by $R_L = 0.4q^{0.3}a$ and it obeys the mass radius relation $R_1 = KM_1^{\beta}$, where β is a constant and K changes slowly due to stellar evolution. Show that mass transfer will occur on a more rapid time scale if

$$q > \frac{17 + 10\beta}{23}.$$

The system now loses angular momentum through a stellar wind such that

$$\frac{dJ}{dt} = fJ\frac{\dot{M}_1}{M_1},$$

where f is a constant. The mass carried away by the wind is negligible. Show that the condition for rapid mass transfer now becomes

$$q > \frac{17 - 20f + 10\beta}{23}.$$

Taking M_1 to be in the solar mass range with $\beta = 0.5$, what range of q would you expect to yield rapid mass transfer?

(You may assume that the angular velocity, Ω, of a binary with a circular orbit and separation, a, is given by $\Omega^2 = G(M_1 + M_2)/a^3$.)

END OF PAPER