UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 12.00

PAPER 65

ASTROPHYSICAL FLUID DYNAMICS

*Attempt **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

Candidates are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}\,,$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}\,,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}\,,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})\,,$$

$$\nabla \cdot \mathbf{B} = 0\,,$$

$$\nabla^2 \Phi = 4\pi G \rho\,.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A supernova explosion of energy E occurs at time $t = 0$ in an unmagnetized polytropic ideal gas of adiabatic exponent γ. The surrounding medium is initially cold and has non-uniform density $Cr^{-\beta}$, where C and β are constants (with $0 < \beta < 3$) and r is the distance from the supernova.

(a) Explain why a self-similar spherical blast wave may be expected to occur, and deduce that the radius $R(t)$ of the shock front increases as a certain power of t.

(b) Write down the self-similar form of the velocity, density and pressure for $0 < r < R(t)$ in terms of three undetermined dimensionless functions of $\xi = r/R(t)$. Obtain a system of dimensionless ordinary differential equations governing these functions.

(c) Formulate the boundary conditions on the dimensionless functions at the strong shock front $\xi = 1$. [You may assume that the solutions of the Rankine–Hugoniot relations in the rest frame of a stationary normal shock are

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}\,,$$

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}\,,$$

$$\mathcal{M}_2^2 = \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{2\gamma\mathcal{M}_1^2 - (\gamma-1)}\,,$$

where $\mathcal{M} = u/v_{\mathrm{s}}$ is the Mach number.]

(d) Show that special solutions exist in which the radial velocity and the density are proportional to r for $r < R(t)$, if

$$\beta = \frac{7-\gamma}{\gamma+1}\,.$$

For the case $\gamma = 5/3$ express the velocity, density and pressure for this special solution in terms of the original dimensional variables.

2 (a) Derive the expressions

$$v^2 = v_{\mathrm{a}}^2 \cos^2\theta \,,$$
$$v^2 = \tfrac{1}{2}(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2) \pm \left[\tfrac{1}{4}(v_{\mathrm{s}}^2 + v_{\mathrm{a}}^2)^2 - v_{\mathrm{s}}^2 v_{\mathrm{a}}^2 \cos^2\theta\right]^{1/2} \,,$$

for the phase speeds of the Alfvén and magnetoacoustic waves in a homogeneous fluid with a uniform magnetic field, explaining the notation used.

(b) Obtain approximate expressions for the magnetoacoustic wave speeds in the limit $v_{\mathrm{s}} \gg v_{\mathrm{a}}$, and describe the physical nature of the three wave modes in this limit. [An expansion to first order in the small parameter $v_{\mathrm{a}}^2/v_{\mathrm{s}}^2$ is sufficient.]

(c) Investigate whether either of the following is an exact nonlinear solution of the equations of ideal MHD in a compressible fluid, where a, k and v_{a} are constants:

(i) a linearly polarized Alfvén wave with

$$B_x = aB_z \cos[k(z - v_{\mathrm{a}}t)] \,,$$
$$B_y = 0 \,,$$
$$B_z = \text{constant} \,;$$

(ii) a circularly polarized Alfvén wave with

$$B_x = aB_z \cos[k(z - v_{\mathrm{a}}t)] \,,$$
$$B_y = aB_z \sin[k(z - v_{\mathrm{a}}t)] \,,$$
$$B_z = \text{constant} \,.$$

3 (a) An ideal polytropic gas undergoes a steady axisymmetric outflow in the presence of a magnetic field and a gravitational potential Φ. Using the representation $\mathbf{B}_{\rm p} = \nabla\psi \times \nabla\phi$ of the poloidal magnetic field in terms of a flux function $\psi(R, z)$, where (R, ϕ, z) are cylindrical polar coordinates, derive the following integrals of the outflow:

$$\mathbf{u} = \frac{k(\psi)\mathbf{B}}{\rho} + R\omega(\psi)\,\mathbf{e}_\phi\,,$$

$$u_\phi - \frac{B_\phi}{\mu_0 k(\psi)} = \frac{\ell(\psi)}{R}\,,$$

$$s = s(\psi)\,,$$

where s is the specific entropy.

(b) Without giving a formal derivation, explain physically why there is a further integral of the form

$$\tfrac{1}{2}|\mathbf{u} - R\omega(\psi)\,\mathbf{e}_\phi|^2 + w + \Phi - \tfrac{1}{2}[R\omega(\psi)]^2 = \varepsilon(\psi)\,,$$

where w is the specific enthalpy.

(c) Consider a particular magnetic field line that lies in the plane $z = 0$ and for which the poloidal magnetic field strength varies as $|\mathbf{B}_{\rm p}| = CR^{-2}$, where C is a constant. Assume that the gravitational potential is that of a point mass M and that the enthalpy is negligible. Assume further that the outflow passes smoothly through an Alfvén point at $R = R_{\rm a}$ where the density is $\rho_{\rm a}$. Deduce that the integrals of the outflow can be combined in the dimensionless equation

$$f(x, y) = \text{constant}\,,$$

where $x = R/R_{\rm a}$, $y = \rho/\rho_{\rm a}$,

$$f(x, y) = \frac{\alpha}{2}\left[\left(\frac{x - x^{-1}}{y - 1}\right)^2 - x^2\right] + \frac{\beta}{2x^4y^2} - \frac{1}{x}\,,$$

and α and β are constants to be determined. Without a detailed calculation, state the form of the conditions for the flow to pass smoothly through the slow and fast magnetosonic points.

4 An isothermal ideal gas of sound speed c_{s} forms a self-gravitating slab in hydrostatic equilibrium with density $\rho(z)$, where (x, y, z) are Cartesian coordinates.

(a) Verify that

$$\rho \propto \operatorname{sech}^2\left(\frac{z}{H}\right),$$

and relate the scaleheight H to the surface density

$$\Sigma = \int_{-\infty}^{\infty} \rho \, \mathrm{d}z \, .$$

(b) Assuming that the perturbations are also isothermal, derive the linearized equations governing displacements of the form

$$\mathrm{Re}\left[\boldsymbol{\xi}(z)\, \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}\right],$$

where k is a real wavenumber. Show that ω^2 is real for disturbances satisfying appropriate conditions as $|z| \to \infty$.

(c) For a marginally stable mode with $\omega^2 = 0$, derive the associated Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(1-\tau^2)\frac{\mathrm{d}\Phi'}{\mathrm{d}\tau}\right] + \left[2 - \frac{\nu^2}{(1-\tau^2)}\right]\Phi' = 0\,,$$

where $\tau = \tanh(z/H)$, $\nu = kH$ and Φ' is the Eulerian perturbation of the gravitational potential. Verify that two solutions of this equation are

$$\left(\frac{1+\tau}{1-\tau}\right)^{\nu/2}(\nu - \tau) \qquad \text{and} \qquad \left(\frac{1-\tau}{1+\tau}\right)^{\nu/2}(\nu + \tau)\,.$$

Deduce that the marginally stable mode has $|k| = 1/H$ and $\Phi' \propto \operatorname{sech}(z/H)$. Would you expect the unstable modes to have wavelengths greater or less than $2\pi H$?

END OF PAPER