UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 1.30 to 3.30

PAPER 64

ADVANCED COSMOLOGY

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) In the 3+1 formalism, we split spacetime using the line element

$$ds^2 = -N^2dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,,$$

with lapse function $N(t,,x^i)$, shift vector $N^i(t,,x^i)$ and the three-metric ${}^{(3)}g_{ij}(x^i)$ on constant time spacelike hypersurfaces Σ. (Latin indices vary over 1,2,3.)

The four-vector $n^\mu = \frac{1}{N}(1,\ N^i)$ is normal to Σ and defines the extrinsic curvature $K_{ij} \equiv -n_{i;j}$. Show that the extrinsic curvature can be expressed as

$$K_{ij} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)\,,$$

where $|$ denotes the covariant derivative in Σ and you may assume that the connection is defined by $\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}\left(g_{\nu\kappa,\lambda} + g_{\lambda\kappa,\nu} - g_{\nu\lambda,\kappa}\right)$.

(ii) When linearising the 3+1 metric about a flat FRW universe, we define the scalar perturbations by

$$N(t,x^i) \equiv \bar{N}(t)(1+\Phi(t,x^i))\,, \qquad N_i \equiv -a^2 B_{,i}\,, \qquad {}^{(3)}g_{ij} = a^2[(1-2\Psi)\delta_{ij} - 2E_{,ij}]\,,$$

$\rho = \bar{\rho} + \delta\rho$ and $P = \bar{P} + \delta P$, where bars denote background homogeneous quantities. In synchronous gauge, we take $\Phi = 0$ and $B = 0$. Given that metric perturbations transform as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \bar{g}_{\mu\nu,0}\xi^0 - \bar{g}_{\lambda\nu}\xi^\lambda_{,\mu} - \bar{g}_{\mu\lambda}\xi^\lambda_{,\nu}$$

under

$$(t,\ x^i) \longrightarrow (\tilde{t},\ \tilde{x}^i) = (t+\xi^0,\ x^i+\xi^i)\,, \quad (\xi^i \equiv \partial^i\lambda)\,,$$

show that there is a residual gauge freedom in synchronous gauge given by the coordinate transformation,

$$\xi^0 = \frac{C(x^i)}{\bar{N}}\,, \qquad \lambda = C(x^i)\int\frac{\bar{N}}{a^2}dt + D(x^i)\,,$$

where C and D are arbitrary functions of x^i only.

In longitudinal Newtonian gauge we take instead $E = B = 0$. Find a transformation law that expresses the density perturbation $\delta\rho/\rho$ in Newtonian gauge in terms of synchronous gauge quantitites.

(iii) Prove that the quantity

$$\zeta = -\Psi + \frac{1}{3}\frac{\delta\rho}{\bar{\rho}+\bar{P}}$$

is gauge-invariant. Show that ζ is independent of time on superhorizon scales, that is, $\dot{\zeta} = 0$ for $k \ll aH$. Briefly discuss the importance of the perturbation variable ζ in inflationary scenarios.

[*Hint:* You may assume a definite equation of state $P = w\rho$, that the perturbed energy density conservation equation is

$$\dot{\delta\rho}/\bar{N} = -3H(\delta\rho + \delta P) + (\bar{\rho}+\bar{P})(\kappa - 3H\Phi) - \triangle u\,,$$

and that the metric perturbation Ψ satisfies $\dot{\Psi}/\bar{N} = -H\Phi + \frac{1}{3}\kappa + \frac{1}{3}\triangle\chi$, where $\triangle \equiv \nabla^2/a^2$, u generates the scalar velocity perturbation, and κ and χ generate the trace and traceless part of K_{ij} respectively.]

2 In a flat FRW universe ($\Omega = 1$), in synchronous gauge (specifying metric perturbations with $h^{0\mu} = 0$), the perturbations of a multicomponent fluid obey the following evolution equations

$$\begin{aligned} &\delta_N' + (1+w_N)i\mathbf{k}\cdot\mathbf{v}_N + \tfrac{1}{2}(1+w_N)h' = 0\,, \\ &\mathbf{v}_N' + (1-3w_N)\frac{a'}{a}\mathbf{v}_N + \frac{w_N}{1+w_N}i\mathbf{k}\delta_N = 0\,, \\ &h'' + \frac{a'}{a}h' + 3\left(\frac{a'}{a}\right)^2 \sum_N (1+3w_N)\Omega_N\delta_N = 0\,, \end{aligned} \qquad (\dagger)$$

where δ_N is the density perturbation, Ω_N is the fractional density, $\mathbf{v}_N$ is the velocity and $P_N = w_N\rho_N$ is the equation of state of the Nth fluid component, and $\mathbf{k}$ is the comoving wavevector ($k = |\mathbf{k}|$), h is the trace of the metric perturbation and primes denote differentiation with respect to conformal time τ ($d\tau = dt/a$).

(i) Suppose that the universe is composed of two components, (comoving) cold dark matter ρ_C with no pressure ($P_C = 0$) and a radiation fluid ρ_R with equation of state $P_R = \rho_R/3$. Show that the coupled matter-radiation equations arising from ($\dagger$) become

$$\begin{aligned} &\delta_C'' + \frac{a'}{a}\delta_C' - \frac{3}{2}\left(\frac{a'}{a}\right)^2 (\Omega_C\delta_C + 2\Omega_R\delta_R) = 0\,, \\ &\delta_R'' + \frac{1}{3}k^2\delta_R - \frac{4}{3}\delta_C'' = 0\,. \end{aligned}$$

Consider times well before equal matter-radiation (i.e. $\tau \ll \tau_{\rm eq}$ when $\rho_R = \rho_C$), to find approximate growing mode solutions for matter and radiation density perturbations which are initially adiabatic:

$$\begin{aligned} &\delta_C = A\tau^2 = \frac{3}{4}\delta_R\,, \quad \text{for} \quad \tau \ll 2\pi/k\,, \\ &\delta_C \approx B\ln\tau\,, \qquad \delta_R \approx C\cos(k\tau/\sqrt{3}) + D\sin(k\tau/\sqrt{3})\,, \quad \text{for} \quad \tau \gg 2\pi/k\,, \end{aligned}$$

where A, B, C, D are functions of the wavevector $\mathbf{k}$ only. Briefly comment on the implications of these solutions for large-scale structure formation.

(ii) Now consider another flat FRW model in which the late universe is dominated by a non-relativistic fluid component $\rho_{\rm m}$ well after matter-radiation equality at $t_{\rm eq}$. With the non-relativistic pressure satisfying $P_m = w_m\rho_m \ll \rho_m$ (w_m const.), use the evolution equations ($\dagger$) to derive the perturbation equation for δ_m:

$$\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m - \left[4\pi G\bar{\rho}_m - c_s^2k^2/a^2\right]\delta_m = 0\,, \qquad (\ddagger)$$

where the sound speed is $c_{\rm s}^2 \equiv dP/d\rho$, here with $w_m = c_s^2$, and dots denote differentiation with respect to cosmic time t.

Assume that this perturbation equation ($\ddagger$) is also valid for a polytropic fluid with an equation of state $P_m \propto \rho_m^{4/3}$, that is, for a non-constant sound speed c_s^2. Find explicit growing and decaying solutions for the density perturbation δ_m in the matter era $t \gg t_{\rm eq}$. Define the Jeans length $\lambda_{\rm J}$ for this fluid and use it to interpret the behaviour of your growing mode solution in different wavelength regimes.

3 (i) Consider a photon with four-momentum p^μ ($p_\mu p^\mu = 0$) propagating in a perturbed FRW universe (flat $\Omega = 1$) with line element

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] ,$$

where $\mathbf{k}$ is the comoving wavenumber and $\hat{k}^i = k^i/|\mathbf{k}|$. A comoving observer with four-velocity $u^\mu = a^{-1}(1,\, 0,\, 0,\, 0)$ measures the photon energy to be $E = -u_\mu p^\mu = ap^0 \equiv q/a$ where q is the comoving momentum. Use the geodesic equation $\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}p^\nu p^\sigma = 0$ to show that along a photon trajectory in (unit) direction $\hat{n}^i$ we have to linear order

$$\frac{dq}{d\tau} = -\frac{1}{2}qh'_{ij}\hat{n}^i\hat{n}^j\,, \qquad\qquad \frac{d\hat{n}^i}{d\tau} = \mathcal{O}(h_{ij})\,.$$

[*Hint:* You may assume that $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^0_{0i} = 0$, $\Gamma^0_{ij} = \frac{a'}{a}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}$, $\Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}$ and $\Gamma^i_{jk} = \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i})$.]

(ii) Assume that the photon brightness function $\Delta(x^i,\, \hat{n}^i,\, \tau) \equiv 4\Delta T/T$ satisfies the collisionless Boltzmann equation which in Fourier space is given by

$$\Delta' + ik\mu\Delta = -2h'_{ij}\hat{n}^i\hat{n}^j = -\tfrac{4}{3}\left[\tfrac{1}{2}h' + \tfrac{1}{2}(3\mu^2 - 1)h'_{\mathrm{s}}\right] , \qquad (*)$$

where $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$ and h and h_{s} are the scalar trace and anisotropic scalar metric perturbations respectively.

Argue that if the photon fluid is in equilibrium for $\tau \le \tau_{\mathrm{dec}}$, we may approximate its initial conditions at photon decoupling by

$$\Delta(\mathbf{k},\, \mu,\, \tau_{\mathrm{dec}}) = \delta_\gamma(\tau_{\mathrm{dec}}) + 4\mathbf{n}\cdot\mathbf{v}(\tau_{\mathrm{dec}})\,,$$

that is, briefly justify why the higher order moments $\Delta_\ell \approx 0$ ($\ell \ge 2$) can be neglected.

Hence, assuming instantaneous decoupling, integrate $(*)$ from decoupling τ_{dec} to today τ_0 to find the Sachs-Wolfe formula for the CMB temperature anisotropy seen at position $\mathbf{x}$ in a direction $\mathbf{n}$:

$$\frac{\Delta T}{T}(\mathbf{x}, \mathbf{n}, \tau_0) = \frac{1}{4}\delta_\gamma(\tau_{\mathrm{dec}}) + \mathbf{n}\cdot\mathbf{v}(\tau_{\mathrm{dec}}) - \frac{1}{2}\int_{\tau_{\mathrm{dec}}}^{\tau_0} d\tau h'_{ij}\;\hat{n}^i\hat{n}^j\,. \qquad (\dagger)$$

Explain the meaning of each term in the formula (†), and describe the length scales on which these contributions are important.

(iii) In Fourier space, integrate the Sachs-Wolfe formula (†) by parts (using the right hand side of $(*)$) to bring it to the following form:

$$\frac{\Delta T}{T}(\mathbf{k}, \mu, \tau_0) = \left[\frac{1}{4}\delta_\gamma + \frac{3i\mu}{4k}\delta'_\gamma - \frac{i\mu}{2k}(h' - h'_{\mathrm{s}}) - \frac{h''_{\mathrm{s}}}{2k^2}\right] e^{-ik\mu(\tau_0 - \tau_{\mathrm{dec}})}$$
$$- \frac{1}{2}\int_{\tau_{\mathrm{dec}}}^{\tau_0} d\tau\; e^{ik\mu(\tau - \tau_0)}\left[\frac{1}{6}(h' - h'_{\mathrm{s}}) - \frac{h'''_{\mathrm{s}}}{2k^2}\right] .$$

[You may assume the equation for the photon density perturbation $\delta'_\gamma + \frac{4}{3}i\mathbf{k}\cdot\mathbf{v} + \frac{2}{3}h' = 0$.]

END OF PAPER